A.M.2 - LEZ. 3 - INTEGRALE DI RIEMANN (3)

0) $\mathcal{R}([a,b])$ È SP. VETT. E $\int$ È LINEARE. (finire)

1) $f, g \in \mathcal{R}([a,b]) \Rightarrow f^+, f^-, |f|, \max\{f,g\}, \min\{f,g\} \in \mathcal{R}([a,b])$

2) MONOTONIA

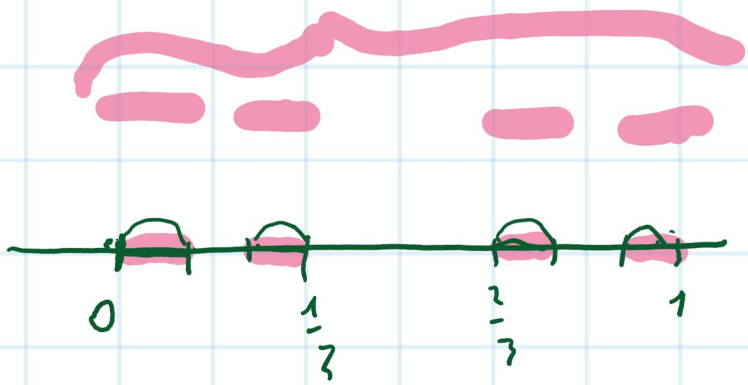
3) $\boxed{\left|\int f\right| \leq \int |f|}$

4) ADDITIVITÀ SU INTERVALLI

5) $\int$ ORIENTATO DEF. E ADDITIVITÀ

6) T.F.C.I + LIPSCH

7) CONSEGUENZE



$C = \{ x \in [0,1] \mid$ SCRITTI IN BASE 3 LA PARTE DECIMALE NON CONTIENE "1" $\}$

0, ...

$\boxed{T.}$ DATI $[a,b]\subset\mathbb{R}$, $\alpha\in\mathbb{R}$, $f,g\in\mathcal{R}([a,b])$

ALLORA

1) $\alpha f\in\mathcal{R}([a,b])$ E $\int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx$

2) $f+g\in\mathcal{R}([a,b])$ E $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx+\int_a^b g(x)\,dx$

$\boxed{\text{DIM}}$ (2) $P=\{x_0,\dots,x_n\}$ PART. DI $[a,b]$

$$\boxed{s(f+g,P)} = \sum_{i=1}^{n}(x_i-x_{i-1})\cdot\inf\{f(x)+g(x)\mid x\in[x_{i-1},x_i)\}\geq$$

$$\boxed{\geq}\sum_{i=1}^{n}(x_i-x_{i-1})\Big(\overbrace{\inf_{x\in[x_{i-1},x_i)} f(x)}^{\lambda}+\overbrace{\inf_{x\in[x_{i-1},x_i)} g(x)}^{\mu}\Big)=$$

$$=\sum_{i=1}^{n}(x_i-x_{i-1})\inf_{x\in I_i} f(x)+\sum_{i=1}^{n}(x_i-x_{i-1})\inf_{x\in I_i} g(x)=$$

$$=\underline{s(f,P)+s(g,P)}$$

$\forall x\in[x_{i-1},x_i)$

$\lambda\le f(x)\ \forall x\in I_i$

$\mu\le g(x)\ \forall x\in I_i$

$\lambda+\mu\le \underline{f(x)+g(x)}\ \forall x\in I_i$

$\lambda+\mu\le \inf_{x\in I_i}(f(x)+g(x))$

$$S(f+g,P)\le\dots\le S(f,P)+S(g,P)$$

$\forall\varepsilon>0$ PRENDO P_1 T.C. $s(f,P_1)>\int_a^b f(x)\,dx-\frac{\varepsilon}{2}$

PRENDO P_2 T.C. $s(g,P_2)>\int_a^b g(x)\,dx-\frac{\varepsilon}{2}$

$\underline{P}=\underline{P_1\cup P_2}$

$$\overline{\int} f+g \geq s(f+g,P)\geq \underline{s(f,P)+s(g,P)} > \int_a^b f(x)\,dx+\int_a^b g(x)\,dx-\varepsilon$$

$\forall \varepsilon > 0 \qquad \underline{\int} f+g > \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon$

$\rightarrow \underline{\int} f+g \geq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

[. . . .]

$\rightarrow \overline{\int} f+g \leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$\leq \underline{\int} f+g \leq \overline{\int} f+g \leq \boxed{\int_a^b f(x)\,dx + \int_a^b g(x)\,dx}$

$f+g \in \mathcal{R}([a,b])$ E $\int_a^b f(x)+g(x)\,dx =$

$\boxed{\text{T.}}$ DATI $[a,b] \subset \mathbb{R}$ $\quad f,g \in \mathcal{R}([a,b])$ ALLORA

1) $f^+, f^-, |f|$ sono $\mathcal{R}([a,b])$

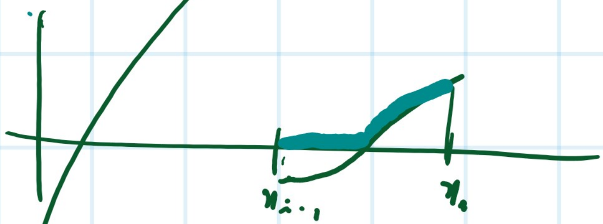
2) $\text{MAX}(f,g)$, $\text{MIN}(f,g) \in \mathcal{R}([a,b])$

3) $f \leq g \Rightarrow \int f \leq \int g$

$\boxed{\text{DIM.}}$ $\quad f^+(x) = \text{MAX}\{f(x), 0\}$

$P = \{x_0, \dots, x_n\}$

$$\boxed{\sum_{i=1}^{n} (x_i - x_{i-1}) \cdot osc\left(f^+, [x_{i-1}, x_i]\right)} \le \sum_{i=1}^{n} (x_i - x_{i-1})\, osc\left(f, [x_{i-1}, x_i]\right)$$



$\forall \varepsilon > 0 \ \exists P$ t.c. $\square < \varepsilon$

$\square < \varepsilon$

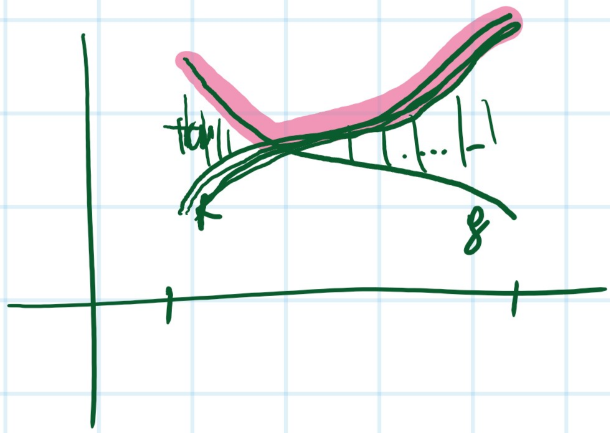
$$f^- = (-f)^+$$

$$f^-(x) = -MIN(f(x), 0) = MAX(-f(x), 0) = (-f)^+$$

$$f(x) = f^+(x) - f^-(x)$$

$$|f(x)| = f^+(x) + f^-(x)$$

$$\boxed{MAX(f(x), g(x))} = \boxed{f(x) + \left(g(x) - f(x)\right)^+}$$



$\boxed{f(x) \ge g(x)}$

$\boxed{f(x) < g(x)}$

$$\underbrace{MAX(f(x), g(x))} = \underbrace{\overbrace{f(x)} + \overbrace{(g(x) - f(x))^+}}$$

$$MIN(f(x), g(x)) = -MAX(-f(x), -g(x))$$

$$\overbrace{f(x) \geq g(x)}$$

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

$$P = \{x_0, \dots, x_n\}$$

$$s(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in I_i} f(x)$$

$$s(g, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in I_i} g(x)$$

$$\int_a^b f \geq \int_a^b g$$

$$\forall P \qquad s(f, P) \geq s(g, P) \qquad \underline{\int} f \geq \underline{\int} g$$

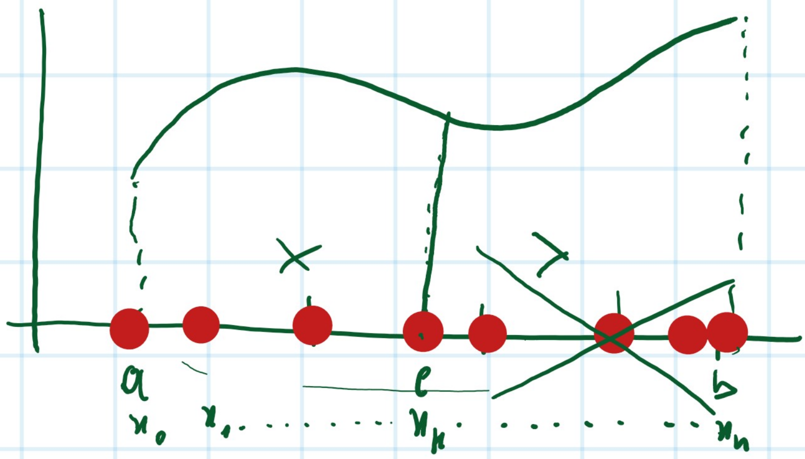
T.1 DATI $[a,c], [c,b] \subset \mathbb{R}$ SIA $f \in \mathcal{R}([a,b])$

ALLORA $f \in \mathcal{R}([a,c])$, $f \in \mathcal{R}([c,b])$ E

$$\boxed{\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx}$$

T.2 $\boxed{}$ SIA $f \in \mathcal{R}([a,c])$ E $f \in \mathcal{R}([c,b])$ ALLORA

$f \in \mathcal{R}([a,b])$ E $\boxed{}$



$\forall \varepsilon > 0$

PRENDO P PART. DI $[a,b]$ T.C. $S(f,P) - s(f,P) < \varepsilon$

$P' = P \cup \{c\}$ VALE $S(f,P') - s(f,P') < \varepsilon$

$P'' = P' - (c, +\infty)$

$$\underbrace{S(f, P'')}- s(f, P'') = < \varepsilon$$

$\boxed{\text{DEF.}}$ DATA $f \in \mathcal{R}([a,b])$ DEFINIAMO

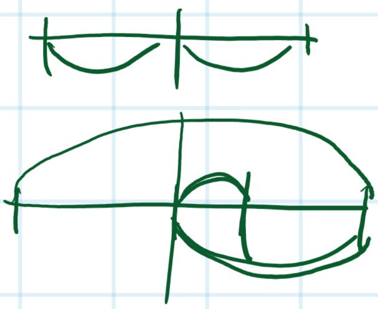
$$\to \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

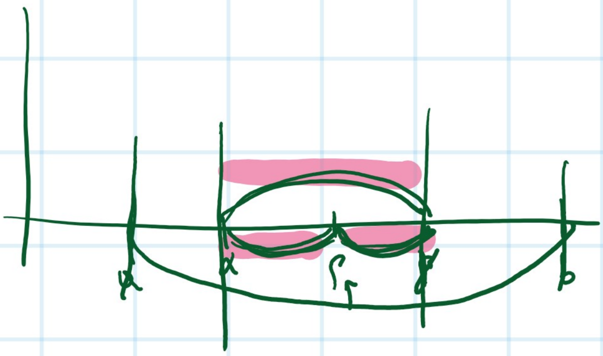
$\boxed{\text{T.}}$ DATI $[a,b] \subset \mathbb{R}$ E $f \in \mathcal{R}([a,b])$

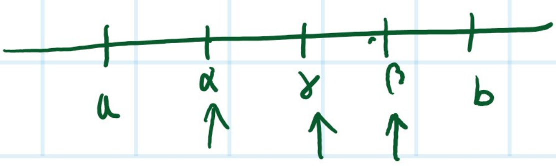
$\forall \alpha, \beta, \gamma \in [a,b]$ SI HA CHE ESISTONO

$$\int_\alpha^\beta f(x)\,dx \;, \int_\beta^\gamma f(x)\,dx \;, \int_\alpha^\gamma f(x)\,dx \quad \text{E}$$

$$\int_\alpha^\beta f(x)\,dx + \int_\beta^\gamma f(x)\,dx = \int_\alpha^\gamma f(x)\,dx$$





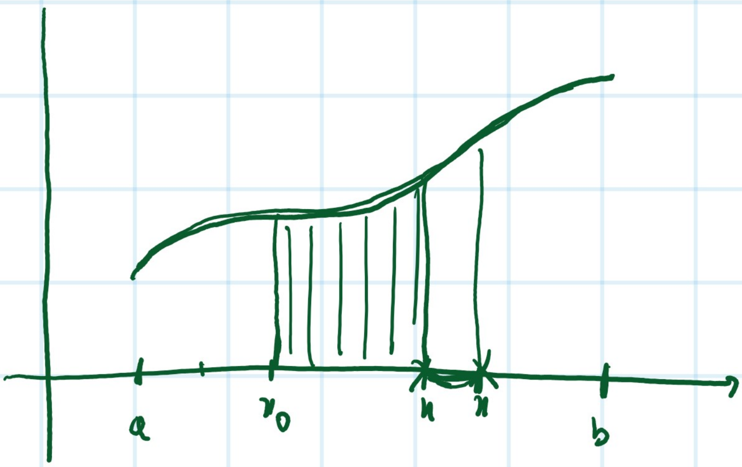


$$\int_\alpha^\beta + \int_\beta^\gamma = \int_\alpha^\gamma \qquad (?)$$

$$-\int_\gamma^\beta$$

$$\int_\alpha^\beta - \int_\gamma^\beta = \int_\alpha^\gamma \qquad (?)$$

$$\int_\alpha^\beta = \int_\alpha^\gamma + \int_\gamma^\beta \qquad (?)$$



$f \in \mathcal{R}([a,b])$

$$F : [a,b] \to \mathbb{R}$$
$$x \longmapsto \int_{x_0}^x f(t)\,dt$$

$$F(x) = \boxed{\int_{x_0}^x f(t)\,dt}$$

T. DATA $f \in \mathcal{R}([a,b])$ SIA $M>0$ T.C. $|f(x)| \leq M$ E SIA

$$F(x) = \int_{x_0}^{x} f(t)\,dt \qquad \forall x \in [a,b] \qquad \text{ALLORA}$$

$$\forall x, y \in [a,b] \qquad |F(x) - F(y)| \leq M\,|x-y|$$

DIM.

$$|F(x)-F(y)| = \left| \int_{x_0}^{x} f(t)\,dt - \int_{x_0}^{y} f(t)\,dt \right| =$$

$$= \left| \int_{x}^{y} f(t)\,dt \right| \leq$$

$$\leq \left| \int_{x}^{y} |f(t)|\,dt \right| \leq$$

$$\leq \left| \int_{x}^{y} M \cdot 1\,dt \right| =$$

$$= M \cdot \left| \int_{x}^{y} 1\,dt \right| =$$

$$= M \cdot |y-x|$$

$$\int_{x}^{y} 1\,dt = \begin{cases} y-x & y>x \\ -\int_{y}^{x} 1\,dt = -(x-y) = y-x & y<x \end{cases}$$

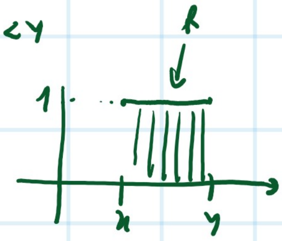
$\forall P$ part.

$s(f,P) = S(f,P) =$

$= x-y$

$\int_{x}^{y} 1\,dt = y-x$

$x<y$



T. (F.C.I) DATI $[a,b] \subset \mathbb{R}$ $f \in \mathcal{R}([a,b])$ $x_0 \in [a,b]$ (CON f continua in $\bar{x} \in [a,b]$)

$$\forall x \in [a,b] \quad F(x) = \int_{x_0}^{x} f(t)\,dt.$$

ALLORA $F(x)$ È DERIV. IN $\bar{x}$ E $F'(\bar{x}) = f(\bar{x})$.

DIM.

$$F'(\bar{x}) = \lim_{h\to 0} \frac{F(\bar{x}+h) - F(\bar{x})}{h} =$$

$$= \lim_{h\to 0} \frac{\int_{x_0}^{\bar{x}+h} f(t)\,dt - \int_{x_0}^{\bar{x}} f(t)\,dt}{h} =$$

$$= \lim_{h\to 0} \left(\frac{1}{h} \int_{\bar{x}}^{\bar{x}+h} f(t)\,dt \right) \leftarrow \circledast = f(\bar{x})$$

$$\int_{\bar{x}}^{\bar{x}+h} \inf\{f(t) \mid t\in[\bar{x},\bar{x}+h]\} \le \int_{\bar{x}}^{\bar{x}+h} f(t)\,dt \le \int_{\bar{x}}^{\bar{x}+h} \sup\{f(t) \mid t\in[\bar{x},\bar{x}+h]\}$$

$$\mathrm{INF}(\dots) \int_{\bar{x}}^{\bar{x}+h} 1\,dt \le G \le \mathrm{SUP}(\dots) \int_{\bar{x}}^{\bar{x}+h} 1\,dt$$

$$\mathrm{INF}\{f \mid \bar{x},\bar{x}+h\} \le \boxed{\circledast} \le \mathrm{SUP}(\dots)$$

f è cont. in $\bar{x}$

$$\downarrow \qquad\qquad \downarrow$$

$$f(\bar{x}) \qquad\qquad f(\bar{x})$$

$$F(x) = \int_a^x f(t)\,dt$$

$$F(b) = \int_a^b f(t)\,dt$$

G primitiva

$$\boxed{G(x)} = F(x) + c$$

$$G(b) - G(a) = F(b) + \cancel{c} - (F(a) + \cancel{c}) =$$

$$= F(b) - \cancel{F(a)} =$$

$$= F(b) = \boxed{\int_a^b f(t)\,dt}$$