A.M.2. - LEZ. 5 — INTEGRAZIONE IN SENSO IMPROPRIO (15/03/2023)

1) ESEMPI INTRODUTTIVI A PARTIRE DA FUNZ. INTEGRALE : $F(x)=\int_0^x e^{-t}\,dt \to \int_0^{+\infty} e^{-x}\,dx$

$F(x)=\int_0^x \frac{1}{\sqrt{1-t}}\,dt \to \int_0^1 \frac{1}{\sqrt{1-x}}\,dx$

2) DEF. E ESEMPI $\left(\int_1^{+\infty}\frac{1}{x}\,dx,\ \int_0^{+\infty}\sin x\,dx\right)$

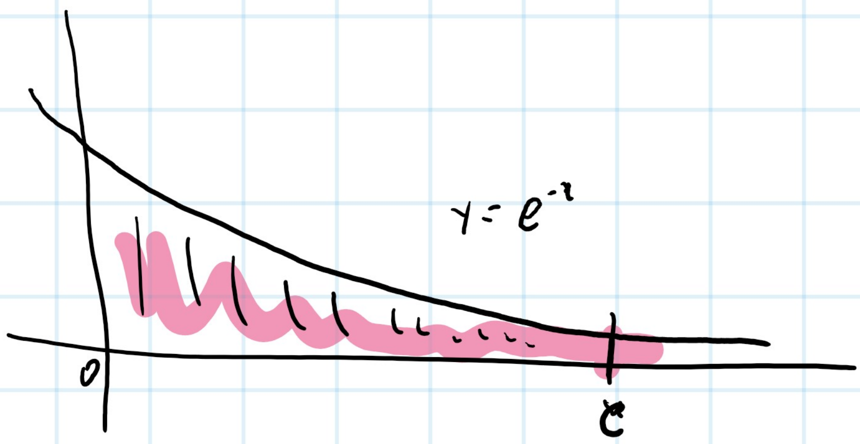
3) INDIPENDENZA DI CARATTERE DA PUNTO BAS

→ 4) $\int_a^b$ IMPROPRIO SIA IN a CHE IN b.

5) ESEMPIO CATTIVO

6) CONV. SE INTEGRANDA ≥ 0 : CRIT. CONF. E CONF. ASINTOTICO DIMO ED ESEMPI $\begin{cases} \int_0^{+\infty} \arctan(e^{-t}+e^{-t^2}\cos t) \\ \int_0^{+\infty} \frac{2+\sin x^2}{1+\sqrt{x}}\,dx \end{cases}$

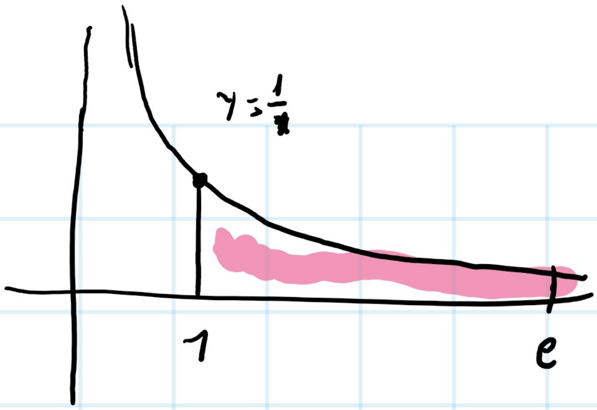
7) INTEGRALI NOTEVOLI



$$F(c)=\int_0^c e^{-t}\,dt$$

$$\int_0^{+\infty} e^{-t}\,dt = \lim_{c\to+\infty}\int_0^c e^{-t}\,dt = \lim_{c\to+\infty}\left[-e^{-t}\right]_0^c =$$

$$= \lim_{c\to+\infty}\left(-\underset{\downarrow\atop 0}{e^{-c}}+e^0\right) = 1$$

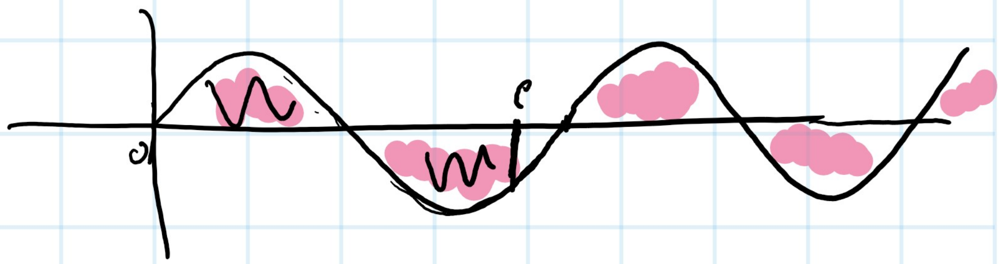


$$\lim_{c\to+\infty}\int_1^c \frac{1}{x}\,dx = \lim_{c\to+\infty}\left[\ln x\right]_1^c = \lim_{c\to+\infty}\left(\ln c - \ln 1\right) =$$

$$= +\infty$$

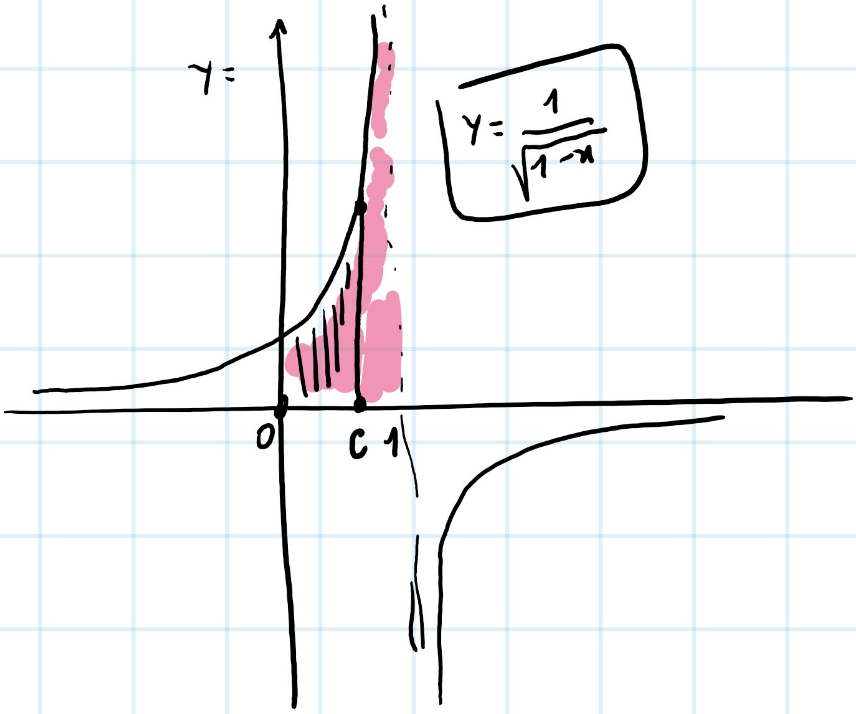
$\int_1^{+\infty} \frac{1}{x}\,dx$ diverge a $+\infty$

$\int$



$$\lim_{c\to+\infty}\int_0^c \sin t\,dt = \lim_{c\to+\infty}\left[-\cos t\right]_0^c =$$

$$= \lim_{c\to+\infty}\left(-\cos c + 1\right) = \text{NON ESISTE}$$



$$\lim_{c \to 1^-} \int_0^c \frac{1}{\sqrt{1-x}}\, dx = \lim_{c \to 1^-} \left[-2\sqrt{1-x} \right]_0^c = \lim_{c \to 1^-} \Big(\underbrace{-2\sqrt{1-c}}_{\downarrow \; 0} + 2\sqrt{1} \Big) = 2$$

$\boxed{\text{DEF.}}$ DATA $f : [a, b) \to \mathbb{R}$ $\left(b \in (a, +\infty] \right)$ T.C. $f \in \mathcal{R}([a, c])$ $\forall c \in (a, b)$ ALLORA DIREMO CHE:

$\int_a^b f(x)\,dx$ converge a $\lambda \in \mathbb{R}$ $\boxed{\text{SE}}$ $\lim_{c \to b^-} \int_a^c f(x)\,dx = \lambda$

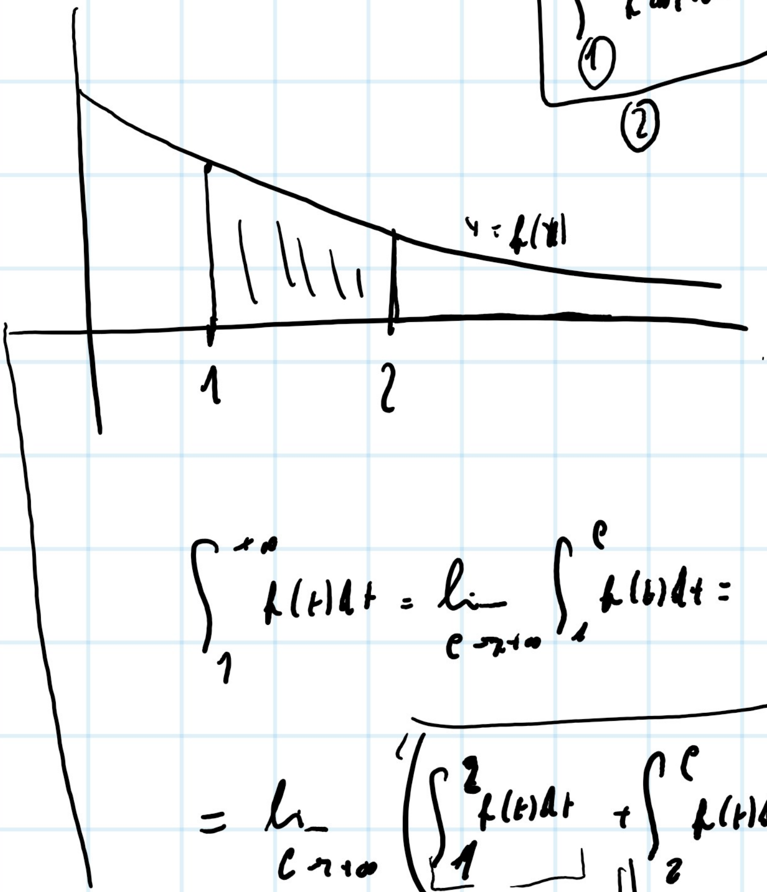
$\int_a^b f(x)\,dx$ diverge a $\overset{(-\infty)}{+\infty}$ $\boxed{\text{SE}}$ $\lim_{c \to b^-} \int_a^c f(x)\,dx = \overset{(-\infty)}{+\infty}$

→ $\int_a^b f(x)\,dx$ è INDETERMINATO $\boxed{\text{SE}}$ $\lim_{c \to b^-} \int_a^c f(x)\,dx$ NON ESISTE.

$$\int_{1}^{+\infty} f(x)\,dx$$

1

2



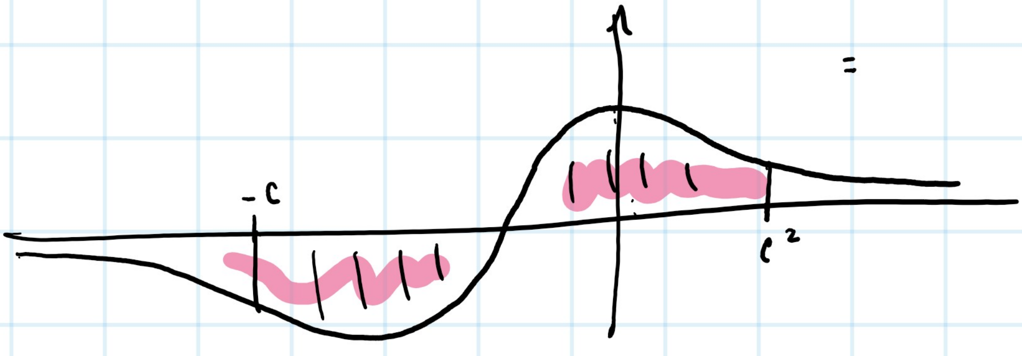
$$\int_1^{+\infty} f(t)\,dt = \lim_{c\to+\infty} \int_1^c f(t)\,dt =$$

$$= \lim_{c\to+\infty}\left(\underbrace{\int_1^2 f(t)\,dt}_{K} + \int_2^c f(t)\,dt\right)$$

NO

$$\int_{-\infty}^{+\infty} \frac{x}{1+x^2}\,dx \qquad \lim_{c\to+\infty}\int_{-c}^{c}\frac{x}{1+x^2}\,dx =$$

$=$



$$\lim_{c\to+\infty} \frac{1}{2}\int_{-c}^{c} \frac{2x}{1+x^2}\,dx = \lim_{c\to+\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_{-c}^{c} = \lim_{c\to+\infty} 0 = 0$$

$$\lim_{c\to+\infty} \int_{-c}^{c^2} \frac{2x}{1+x^2}\,dx = \lim_{c\to+\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_{-c}^{c^2} =$$

$$= \lim_{c\to+\infty} \frac{1}{2}\ln\left(\frac{1+c^4}{1+c^2}\right) = +\infty$$

$$\int_{c}^{c\cdot e}$$

$$\frac{1}{2}\ln\left(\frac{1+2c^2}{1+c^2}\right.$$

$\boxed{\text{DEF.}}$ DATI (a,b) CON $\underline{a<b}$ E TALI CHE $a \in \mathbb{R}\cup\{-\infty\}$ e $b \in \mathbb{R}\cup\{+\infty\}$

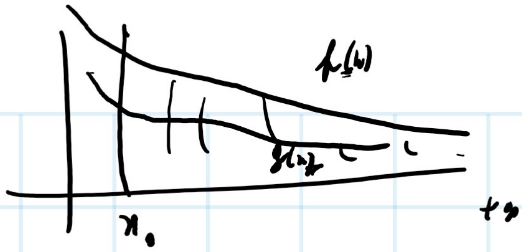
E $x_0 \in (a,b)$ E $f: (a,b) \to \mathbb{R}$ t.c. $f \in \mathcal{R}([\alpha,\beta])$

$\forall \alpha, \beta$ t.c. $a<\alpha<\beta<b$.

DIREMO CHE:

1) $\int_a^b f(x)\,dx$ CONVERGE SE $\int_a^{x_0} f(x)\,dx$ E $\int_{x_0}^b f(x)\,dx$ SONO ENTRAMBI CONV.

2) $\int_a^b f(x)\,dx$ DIVERGE A $\underline{+\infty}$ $(-\infty)$ SE $\int_a^{x_0}$ E $\int_{x_0}^b$ DIVERG. ENTRAMBI $+\infty$ $(-\infty)$ O SE UNO DEI 2 $\overline{\text{DIV.}}$ E L'ALTRA CONV.



$\boxed{T.1}$ $\boxed{\text{CR. CONFRONTO}}$

$\forall x \in [a,b)$

DATI $[a,b) \subset \mathbb{R}$ E $f,g \in \mathcal{R}([a,c])$ $\forall c \in (a,b)$ CON $\boxed{0 \le g(x) \le f(x)}$

ALLORA SI HA:

→ 1) $\boxed{\int_a^b f(x)\,dx \text{ CONV.}} \Rightarrow \boxed{\int_a^b g(x)\,dx \text{ CONV.}}$

→ 2) $\int_a^b g(x)\,dx$ DIV. $\Rightarrow \boxed{\int_a^b f(x)\,dx \text{ DIV.}}$

$\boxed{\text{DIM.}}$

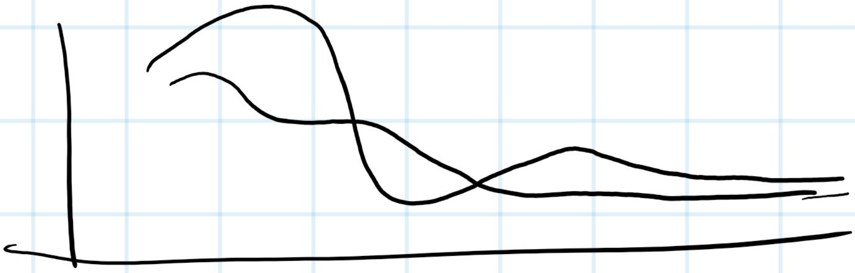
$$\lim_{c \to +\infty} \int_a^c f(x)\,dx = \lambda \in \mathbb{R}$$

$$\boxed{\lim_{c \to +\infty} \int_a^c g(x)\,dx} = \mu \in \mathbb{R} \quad (?)$$

(?)

$$g(x) \le f(x)$$

$$\forall c \in (a,b) \quad \boxed{\int_a^c g(x)\,dx} \le \boxed{\int_a^c f(x)\,dx}$$



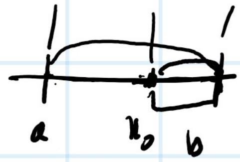
$f, g \in R([a,c])$ $\forall c \in (a,b)$

$\boxed{T.}$ (Conf. Asint.) DATI $[a,b) \subset \mathbb{R}$ ... E $f, g : [a,b) \to (0,+\infty)$

T.C. $f(x) \simeq g(x)$ PER $x \to b^-$ ALLORA

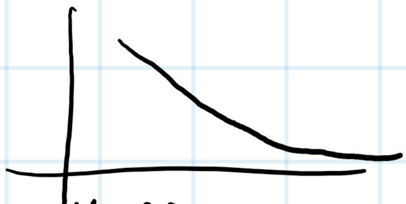
$\int_a^b f(x)\,dx$ E $\int_a^b g(x)\,dx$ HANNO STESSO CARATTERE

$\boxed{\text{DIM}}$ $\boxed{\frac{1}{2} < \frac{f(x)}{g(x)} < \frac{3}{2}}$ SE $x \geq x_0 \in (a,b)$



$$\frac{1}{2} g(x) < f(x) < \frac{3}{2} g(x)$$

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \alpha > 1 & \text{CONV.} \\ \alpha \leq 1 & \text{DIV} \end{cases}$$



$$\int_2^{+\infty} \frac{1}{x^\alpha (\ln x)^\beta}\,dx = \begin{cases} \alpha > 1 & \text{CONV. } \forall \beta \in \mathbb{R} \\ \alpha = 1 & \begin{cases} \beta > 1 & \text{CONV.} \\ \beta \leq 1 & \text{DIV.} \end{cases} \\ \alpha < 1 & \text{DIV. } \forall \beta \in \mathbb{R} \end{cases}$$