A.M.2 - LEZ 11 - SERIE NUMERICHE (II) (3 APRILE 2023)

[-1) DIVERGENZA SERIE ARMONICA

→ 0) ESEMPI (PER C. CONF.): $\sum_{n=1}^{+\infty} \frac{1}{n^2}$, $\sum_{n=1}^{+\infty} \frac{1}{\sqrt[n]{n!}}$, $\boxed{\sum_{n=1}^{+\infty} \frac{2+\cos n}{n}}$ $\boxed{\sum_{n=1}^{+\infty} \frac{1+\cos n}{n}}$

→ 1) CR. CONF. ASINTOTICO

2) ESEMPI $\sum_{n=1}^{+\infty} \sin\frac{1}{n}$, $\sum_{n=1}^{+\infty} \left(\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}\right)$

3) CR. INTEGRALE.

4) ESEMPI NOTEVOLI: $\sum \frac{1}{n^\alpha}$, $\sum_{n=2}^{+\infty} \frac{1}{n^\alpha \ln n^\beta}$

5) ESEMPI $\sum \frac{1}{\ln(1+n^n)}$

6) CRITERI DELLA RADICE E DEL RAPPORTO

7) ESEMPI $\sum \frac{1}{n!}$, $\boxed{\sum \frac{n!\, n^{n+1}}{(2n)!} A^n}$, $\boxed{\sum \frac{n^{2n}}{(2n)!} A^n}$

$$\sum \boxed{\frac{\cancel{1}+\cancel{\cos n}}{n}} = \sum\left(\frac{1}{n} - \frac{\cos n}{n}\right)$$

$$\sum \frac{1}{\sqrt[n]{n!}}$$

$$\boxed{\frac{1}{\sqrt[n]{n!}}} > \frac{1}{\sqrt[n]{n^n}} = \boxed{\frac{1}{n}}$$

$$\sum_{n=1}^{+\infty} \frac{2+\cos n}{n}$$

$$\frac{2+\cos n}{n} \geq \frac{1}{n}$$

$$1 + \underbrace{(1+\cos n)}_{\geq 0}$$

$$\sum \frac{1}{\ln(1+n^n)} \qquad \sum \frac{1}{n \ln n}$$

$$\bigcirc \simeq \frac{1}{n\ln n}$$

$$\ln(1+n^n) \approx \ln(n^n) = n\ln n$$

$$\to \sum_{n=1}^{+\infty} \frac{1}{n}$$

$$\leadsto \boxed{\sum_{n=2^k+1}^{2^{k+1}} \boxed{\frac{1}{n}}} \geq \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}} = \frac{1}{2^{k+1}} \cdot \sum_{n=2^k+1}^{2^{k+1}} 1 = \frac{1}{2^{k+1}} \cdot 2^k = \frac{1}{2}$$

$$\frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \boxed{\frac{1}{2^{k+1}}}$$

$$\forall \varepsilon > 0 \quad \underline{\exists n_0 \in \mathbb{N}} \text{ t.q. } n, m \geq n_0, \ n < m \quad \left| \sum_{i=n+1}^{m} a_i \right| < \varepsilon$$

$$\varepsilon = \frac{1}{2} \qquad \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n} > \frac{1}{2}$$

$$\boxed{\ln(1+x) \leq x} \qquad \forall x > -1$$

$$\to \sum_{n=1}^{+\infty} \frac{1}{n}$$

$$\to \sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right) =$$

$$\boxed{S_n} = \ln\left(1+\frac{1}{1}\right) + \ln\left(1+\frac{1}{2}\right) + \dots + \ln\left(1+\frac{1}{n}\right) =$$

$$= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \dots + \ln\left(\frac{n+1}{n}\right) =$$

$$= \ln\left(\cancel{\frac{2}{1}} \cdot \cancel{\frac{3}{2}} \cdot \cancel{\frac{4}{3}} \cdot \ldots \cdot \frac{n+1}{\cancel{n}}\right) = \boxed{\ln(n+1)}$$

$$\boxed{\sum \boxed{\frac{1}{n^\alpha}} \quad \alpha \geq 2 \quad \text{CONV.}} \qquad \boxed{\sum \frac{1}{n^\alpha} \quad \alpha \leq 1 \quad \text{DIV.}}$$

$$\to \sum_{n=1}^{+\infty} \boxed{\frac{1}{n^2}}$$

$$\to \sum_{n=1}^{+\infty} \frac{2}{n(n+1)}$$

$$\frac{1}{n^2} \leq \frac{2}{n(n+1)} \quad ? \leftarrow$$

$$\frac{1}{n^\alpha} \leq \frac{1}{n}$$

$$\cancel{n}(n+1) \leq 2n^{\cancel{2}} \quad ? \leftarrow$$

$$n+1 \leq 2n \quad ?$$

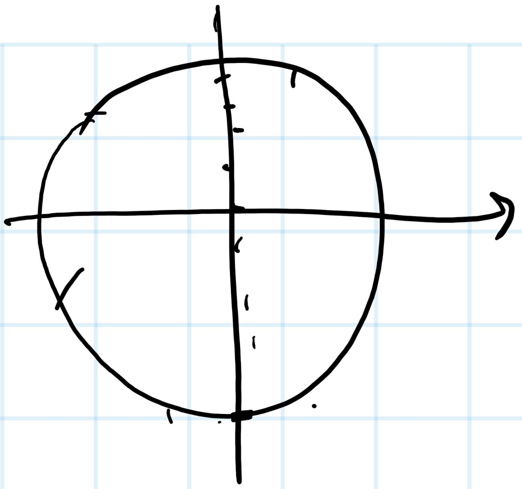
$$1 \leq n \quad ?$$

$$\sum_{n=1}^{+\infty} \frac{\boxed{\frac{11}{10}} + \cos n}{n}$$

$$\frac{\frac{1}{10} + \boxed{(1+\cos n)}}{n} \geq \frac{\frac{1}{10}}{n}$$

$$\sum \frac{\frac{1}{10}}{n} = \frac{1}{10} \sum \frac{1}{n}$$

$$\sum \frac{1+\cos n}{n}$$



$\boxed{1+\cos n}$

$S_n \to +\infty$

$S_{6k} \to +\infty$

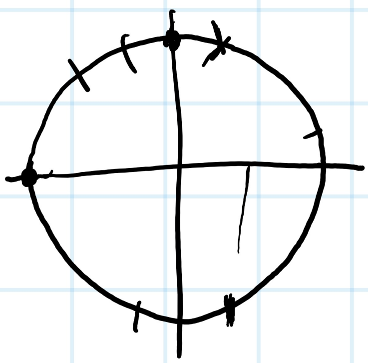
$\frac{1}{6}$ $(\cdot \cdots)$

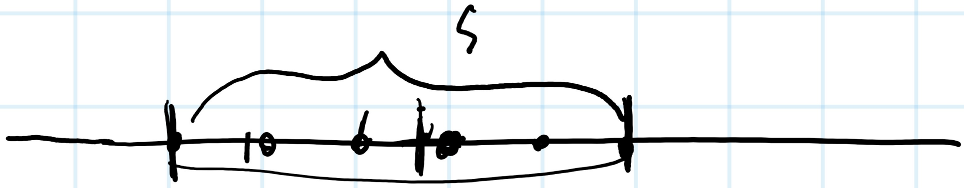
$$(a_1+a_2+\dots+a_6)+\dots+(a_{6k-5}+a_{6k-4}+a_{6k-3}+a_{6k-2}+a_{6k-1}+a_{6k})$$

$$(a_{6i-5}+a_{6i-4}+\dots+a_{6i})$$

$$\geq \frac{1}{6}+\dots+\frac{1}{6k} = \frac{1}{6}\left(1+\dots+\frac{1}{k}\right)$$

$$\frac{1+\cos(6i-5)}{6i-5}+\dots+\frac{1+\cos(6i)}{6i} \geq \boxed{\frac{1}{6i}}$$





$\boxed{k\pi\ \frac{\pi}{7} \qquad k\pi' - \frac{\pi}{2}}$

$$S_{6k} \geq \frac{1}{6}\left(\underbrace{\overbrace{1+\frac{1}{2}+\dots+\frac{1}{k}}}\right) \xrightarrow{k\to+\infty} +\infty$$

$$\boxed{S_{6k} \to +\infty}$$

$$\boxed{S_n \to +\infty} \ ?$$

$\boxed{\text{C.C.AS.}}$ (a_n) (b_n) Termini positivi $\frac{a_n}{b_n} \to 1$

ALLORA $\sum a_n$ E $\sum b_n$ HANNO ST. CARATTERE.

$\boxed{\text{DIM}}$

$$\frac{a_n}{b_n} \to 1$$

DEF. in n $\frac{1}{2} \leq \frac{a_n}{b_n} \leq \frac{3}{2}$

DEF. in n $\underbrace{\frac{1}{2} b_n} \leq \underline{a_n} \leq \frac{3}{2} b_n$

$\boxed{\text{ES.}}$ $\underbrace{\sum_{n=1}^{+\infty}\left(\underbrace{\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}}\right)}$

$$\boxed{\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}} = \boxed{\cancel{\frac{1}{\sqrt{n}}} - \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{6n\sqrt{n}} + o\left(\frac{1}{n^{\frac{3}{2}}}\right)\right)}$$

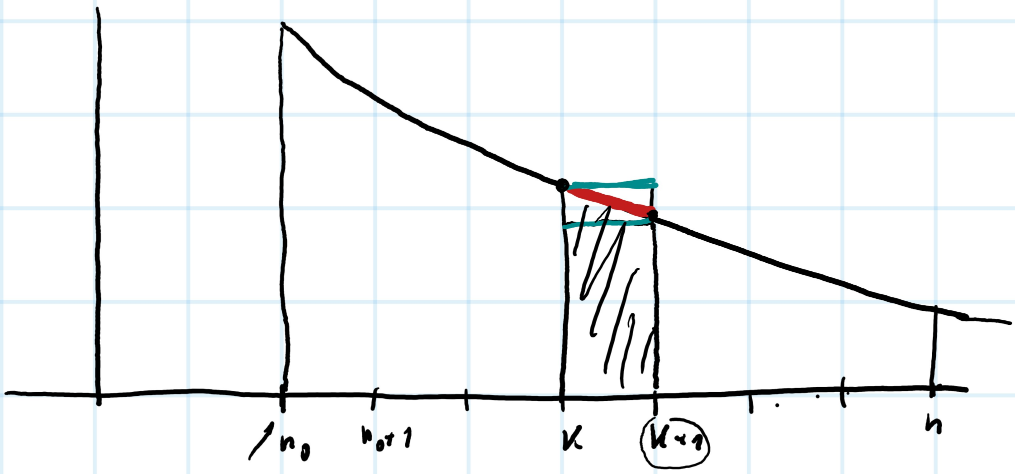
$$= \frac{1}{6n\sqrt{n}} + o\left(\frac{1}{n^{5/2}}\right) = \boxed{\frac{1}{6n\sqrt{n}}}\left(1 + o\left(\frac{1}{n}\right)\right)$$

$$\boxed{\sum \frac{1}{6n\sqrt{n}} = \frac{1}{6}\sum \frac{1}{n^{3/2}}}$$

CR. INT. DATA $f : [n_0, +\infty) \to (0, +\infty)$, $n_0 \in \mathbb{N}$, DECRESCENTE

ALLORA $\int_{n_0}^{+\infty} f(x)\,dx$ $\quad \sum_{n=n_0}^{+\infty} f(n)$ HANNO STESSO CARATTERE

DIM



$$\int_{n_0}^{n} f(x)\,dx = \int_{n_0}^{n_0+1} f(x)\,dx + \int_{n_0+1}^{n_0+2} f(x)\,dx + \dots + \boxed{\int_{k}^{k+1} f(x)\,dx} + \dots + \int_{n-1}^{n} f(x)\,dx =$$

$$f(k+1) = \int_k^{k+1} f(k+1)\,dx \le \boxed{\bullet} \le \int_k^{k+1} f(k)\,dx = \boxed{f(k)}$$

$$S_n - f(n_0) = \boxed{f(n_0+1) + f(n_0+2) + \dots + f(k+1) + \dots + f(n)}$$

$$S_{n-1} = \boxed{f(n_0) + f(n_0+1) + \dots + f(k) + \dots + f(n-1)}$$

$$S_n - f(n_0) \le \int_{n_0}^{n} f(x)\,dx \le \boxed{S_{n-1}}$$

$$S_n \xrightarrow{n\to+\infty}$$

$$S_n - f(n_0) \le \boxed{\int_{n_0}^{n} f(x)\,dx} \le S_{n-1}$$

ES.

$$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \qquad \int_1^{+\infty} \frac{1}{x^\alpha}\,dx$$

$$\sum_{n=2}^{+\infty} \frac{1}{n^\alpha (\ln n)^\beta} \qquad \int_2^{+\infty} \frac{1}{x^\alpha (\ln x)^\beta}\,dx$$

$$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \quad \begin{cases} \alpha > 1 & \text{CONV.} \\ \alpha \le 1 & \text{DIV.} \end{cases}$$

$$\sum_{n=2}^{+\infty} \frac{1}{n^\alpha (\ln n)^\beta} \quad \begin{cases} \alpha > 1 & \text{CONV.} \\ \alpha = 1 & \begin{cases} \beta > 1 & \text{CONV.} \\ \beta \le 1 & \text{DIV.} \end{cases} \\ \alpha < 1 & \text{DIV} \end{cases}$$

$\boxed{\text{C. RADICE}}$ DATA (a_n) a termini positivi

A) $\rightarrow$ SE $\sqrt[n]{a_n} \leq \ell < 1$ DEF. IN n. ALLORA $\sum a_n$ CONV.

B) $\Rightarrow$ SE $\sqrt[n]{a_n} \geq 1$ FREQ. ALLORA $\sum a_n$ DIV.

$\boxed{\text{DIM.}}$

B) $\sqrt[n]{a_n} \geq 1$ FREQ. $\Rightarrow a_n \geq 1$ FREQ. $\Rightarrow a_n \not\to 0 \Rightarrow$

$\Rightarrow \sum a_n$ NON CONV. $\Rightarrow \sum a_n$ DIV.

A) DEF. IN n $\sqrt[n]{a_n} \leq \ell < 1$

$\Updownarrow$

$$\underbrace{\boxed{a_n} \leq \ell^n}$$ ← Serie geom. CONV.

$\Downarrow$

$\sum a_n$ CONV. PER CONFRONTO

$\boxed{\text{CR. RAPP.}}$ DATA (a_n) A TERM. strett. POSITIVI SI HA:

A) SE $\underbrace{\text{DEF. IN } n.}$ $\underbrace{\frac{a_{n+1}}{a_n} \leq \ell < 1}$ ALLORA $\sum a_n$ CONV.

B) SE $\underbrace{\text{DEF. IN } n \ \frac{a_{n+1}}{a_n} \geq 1}$ ALLORA $\sum a_n$ DIV.

$\boxed{\text{DIM}}$

(B) $\Rightarrow$ DEF. IN n $a_{n+1} \geq a_n \Rightarrow \exists n_0 \in \mathbb{N}$ t.c. $a_{n+1} \geq a_n \ \forall n \geq n_0$

$\Rightarrow \forall n \geq n_0 \quad a_n \geq \boxed{a_{n_0}} > 0 \Rightarrow a_n \not\to 0 \Rightarrow \sum a_n$ NON CONV.

$\Rightarrow \sum a_n$ DIV.

(A) POSSO SUPPORRE CHE $\boxed{\frac{a_{n+1}}{a_n} \leq l} < 1$ VALGA $\forall n \in \mathbb{N}$ (PERCHÉ [...])

$$\boxed{a_{n+1} \leq l\, a_n \quad \forall n}$$

$$a_1 \leq l\, a_0$$

$$a_2 \leq l\, a_1 \leq l \cdot l \cdot a_0 = l^2 a_0$$

$$\vdots$$

$$a_k \leq l^k a_0$$

$$a_{k+1} \leq l \cdot a_k \leq l \cdot l^k a_0 = l^{k+1} a_0$$

$\forall n \in \mathbb{N}$ $\boxed{a_n \leq a_0 \cdot l^n}$ $\quad \boxed{\sum a_0 l^n} = a_0 \boxed{\sum l^n}$ SERIE GEO. CONV.

$\Downarrow$

$\sum a_n$ CONV. PER CONFRONTO.

$$\sum_{n=1}^{+\infty} \frac{n^{2n}}{(2n)!} A^n$$

ST. $\forall A > 0$ AL VARIARE DI

$$\frac{(n+1)^{2(n+1)} \cancel{A^{n+1}}}{\cancel{(2n+2)!}} \cdot \frac{\cancel{(2n)!}}{n^{2n} \cdot \cancel{A^n}} = A \cdot \frac{(n+1)^{\cancel{2}}}{\cancel{(2n+2)}_{2}(2n+1)} \cdot \frac{(n+1)^{2n}}{n^{2n}} =$$

$(2n+2)(2n+1)$

$$= A \cdot \boxed{\frac{n+1}{2(2n+1)}} \cdot \left(1 + \frac{1}{n}\right)^{2n} \Rightarrow A \cdot \frac{e^2}{4}$$

$\frac{n+1}{2(2n+1)} \to \frac{1}{4}$

$A > \frac{4}{e^2}$ DIV.

$A = \frac{4}{e^2}$

$A < \frac{4}{e^2}$ CONV.

$2n$

$2(n+1)$

$2n+2$