A.M.2 — SERIE NUMERICHE (III) (4 APRILE 2023)

0) ESEMPI SU CR. RAPP. $\sum \frac{1}{n!}$ $\underbrace{\sum \frac{n!\, n^{n+1}}{(2n)!} A^n}$ $\underbrace{\sum \frac{n^{2n}}{(2n)!} A^n}$

→ 1) CR. CONFRONTO DI RAPPORTI

2) CR. GAUSS

3) ALCUNI QUESITI TEORICI :

FINO A QUI → a) REL. TRA C. RAPPORTO E C. RADICE

b) CRIT. CONDENSAZIONE

4) CALCOLO DI ALCUNE SERIE : a) $\sum_{n=0}^{+\infty} \frac{A^n}{n!}$

b) $\sum_{n=0}^{+\infty} \frac{(n^2+5n+2)}{n!}$

c) $\sum_{n=1}^{+\infty} \frac{n}{2^n}$

$$\sum \boxed{\overset{a_n}{\frac{n^{2n}}{(2n)!} A^n}} \qquad \boxed{A>0}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{2n+2} \cdot A^{\cancel{n+1}}}{\underbrace{\cancel{(2n+2)!}}_{2\cancel{(2n+2)}(2n+1)}} \cdot \frac{\cancel{(2n)!}}{n^{2n} \cdot \cancel{A^n}} = A \cdot \frac{n+1}{2(2n+1)} \cdot \left(\frac{n+1}{n}\right)^{2n} =$$

$$= \boxed{A \cdot \frac{n+1}{4n+2} \cdot \left(\underbrace{\left(1+\frac{1}{n}\right)^n}\right)^2} \to A \cdot \frac{e^2}{4}$$

DIV. $\boxed{A > \frac{4}{e^2}}$

$A = \frac{4}{e}$

CON. $\boxed{A < \frac{4}{e^2}}$

$$\frac{1}{4} \cdot \frac{\underbrace{2n+2}}{2n+1} = \frac{1}{4}\left(1+\frac{1}{2n+1}\right)$$

$$A = \frac{4}{e^2}$$

$$\boxed{\frac{4n+4}{4n+2}} \cdot \underbrace{\left(1+\frac{1}{n}\right)^{2n}} \cdot e^{-2} =$$

$$= \left(1+\frac{1}{2n+1}\right) e^{\boxed{2n \ln\left(1+\frac{1}{n}\right) - 2}} =$$

$$2n \ln\left(1+\frac{1}{n}\right) - 2 = 2n \cdot \left(\frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)\right) \cancel{-2} = \boxed{-\frac{1}{n} + O\left(\frac{1}{n^2}\right)}$$

$$= \left(1+\boxed{\frac{1}{2n+1}}\right) \cdot \left(1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) =$$

$$\boxed{\frac{1}{2n+1}} = \frac{1}{2n+1} - \frac{1}{2n} + \frac{1}{2n} = \frac{1}{2n} + \frac{-1}{(2n+1)2n} = \boxed{\frac{1}{2n} + o\left(\frac{1}{n^2}\right)}$$

$$= \left(1 + \frac{1}{2n} + o\left(\frac{1}{n^2}\right)\right)\left(1 - \frac{1}{n} + o\left(\frac{1}{n^2}\right)\right) =$$

$$= 1 - \frac{1}{n} + \frac{1}{2n} + o\left(\frac{1}{n^2}\right) = 1 - \frac{1}{2n} + o\left(\frac{1}{n^2}\right)$$

$A = \frac{4}{e^2}$

$$\frac{a_{n+1}}{a_n} = \cdots = 1 - \frac{1}{2n} + o\left(\frac{1}{n^2}\right) \qquad 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right)$$

$$\boxed{1 - \frac{\frac{1}{2}}{n} + o\left(\frac{1}{n}\right)}$$

$\boxed{\text{CR. CONF. RAPP.}}$ $(a_n), (b_n)$ A TERMINI ST. POS. t.c.

$\frac{\text{DEF. IN } n}{\forall n}$ $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ ALLORA SI HA

1) $\sum b_n$ CONV. $\Rightarrow$ $\sum a_n$ CONV.

2) $\sum a_n$ DIV. $\Rightarrow$ $\sum b_n$ DIV.

$\boxed{\text{DIM.}}$

MOSTRO CHE $\forall n \in \mathbb{N}$ $\underbrace{\boxed{\frac{a_n}{a_0}} \le \frac{b_n}{b_0}}$ (?)

$$\frac{a_n}{a_0} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_{n-2}}{a_{n-3}} \cdot \frac{a_{n-3}}{\cdots} \cdots\cdots \frac{a_2}{a_1} \cdot \frac{a_1}{a_0} \le$$

$$\le \frac{b_n}{\cancel{b_{n-1}}} \cdot \frac{\cancel{b_{n-1}}}{\cancel{b_{n-2}}} \cdot \frac{\cancel{b_{n-2}}}{\cancel{b_{n-3}}} \cdots\cdots \frac{\cancel{b_2}}{\cancel{b_1}} \cdot \frac{\cancel{b_1}}{b_0} =$$

$$= \frac{b_n}{b_0}$$

$\forall n \in \mathbb{N}$ $\quad \frac{a_n}{a_0} \le \frac{b_n}{b_0} \quad \underbrace{a_n \le \boxed{\frac{a_0}{b_0}} \cdot b_n}$

$\to \sum b_n$ CONV. $\Rightarrow \sum \left(\frac{a_0}{b_0}\right) \cdot b_n$ CONV. $\Rightarrow \sum a_n$ CONV.

$$\frac{a_{n+1}}{a_n} = \underbrace{1 - \frac{\frac{1}{2}}{n} + O\left(\frac{1}{n^2}\right)}$$

CERCO (b_n) t.c. $\underbrace{\frac{b_{n+1}}{b_n} \le \frac{a_{n+1}}{a_n}}$ e $\underline{\sum b_n \text{ DIV.}}$

$b_n = \frac{1}{n-1}$ $\quad$ $\sum b_n$ diverge ?

$\sum_{n=2}^{+\infty} \frac{1}{n-1}$ $\quad$ $\sum_{n=1}^{+\infty} \frac{1}{n}$

$$\boxed{\frac{b_{n+1}}{b_n}} = \frac{\frac{1}{n}}{\frac{1}{n-1}} = \frac{n-1}{n} = \boxed{1-\frac{1}{n}}$$

$$\boxed{\frac{b_{n+1}}{b_n}} \overset{?}{=} \boxed{\frac{a_{n+1}}{a_n}}$$

DEF. IMN ?

Svil. in n

$$\boxed{1-\frac{1}{n}} \leq \boxed{1-\frac{1}{2n}+o\left(\frac{1}{n^2}\right)} \quad ?$$

$$\cancel{1}-\frac{1}{n}\cancel{-1}+\frac{1}{2n}+o\left(\frac{1}{n^2}\right) \leq 0 \quad ?$$

$$-\frac{1}{2n}+o\left(\frac{1}{n^2}\right) \leq 0$$

$$\frac{1}{2n}\left(-1+o\left(\frac{1}{n}\right)\right) \leq 0 \quad (?)$$

$$\sum \boxed{\frac{n!\, n^{n+1}}{(2n)!} A^n} \quad (a_n)$$

$A > 0$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(n+1)!}\cdot(n+1)^{n+2}\cdot A^{\cancel{n+1}}}{\cancel{(2n+2)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{n!}\, n^{n+1}\cdot \cancel{A^{n}}} =$$

$$2\cancel{(2n+2)}(2n+1)$$

$$= A \cdot \underbrace{\frac{n+1}{2(2n+1)}} \cdot \left(\frac{n+1}{n}\right)^{n+1} =$$

$$= A \cdot \frac{1}{4} \cdot \frac{2n+2}{2n+1} \cdot \left(1+\frac{1}{n}\right)^{n+1} =$$

$$= \underbrace{A \cdot \frac{1}{4} \cdot \left(1+\frac{1}{2n+1}\right) \cdot \left(1+\frac{1}{n}\right)^{n+1}} \longrightarrow A \cdot \frac{e}{4}$$

$A > \frac{4}{e}$ DIV.

$A < \frac{4}{e}$ CONV.

$$\boxed{A = \frac{4}{e}} \qquad \left(1+\frac{1}{2n+1}\right) \cdot \frac{\left(1+\frac{1}{n}\right)^{n+1}}{e} \to 1$$

$\boxed{\text{CR. GAUSS}}$

DATA (a_n) A TERMINI POSITIVI. T.C.

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right)$$

ALLORA:

→ ① $\alpha < 1 \Rightarrow \sum a_n$ DIVERGE

② $\alpha > 1 \Rightarrow \sum a_n$ CONVERGE

$\boxed{\text{DIM.}}$

① $\boxed{b_n = \frac{1}{n-1}}$

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{n}}{\frac{1}{n-1}} = \frac{n-1}{n} = \boxed{1 - \frac{1}{n}}$$

↑

$\alpha < 1$

$$\frac{a_{n+1}}{a_n} = \boxed{1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right)}$$

(DEF. IN n) $\quad 1 - \frac{1}{n} \leq 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right)$ $\quad$ (?)

$$\frac{\alpha - 1}{n} + o\left(\frac{1}{n}\right) \leq 0 \qquad (?)$$

(DEF. IN n) $\quad \frac{1}{n}\left(\underbrace{(\alpha - 1)}_{<0} + \underbrace{o(1)}_{\to 0}\right) \leq 0$

(DEF IN n) $\quad \boxed{\frac{b_{n+1}}{b_n} \leq \frac{a_{n+1}}{a_n}}$

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(2)

$\beta > 1$

$$b_n = \frac{1}{(n-1)^\beta}$$

$\sum b_n$ CONV.

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n)^\beta}}{\frac{1}{(n-1)^\beta}} = \left(\frac{n-1}{n}\right)^\beta = \left(1 - \frac{1}{n}\right)^\beta = 1 - \frac{\beta}{n} + o\left(\frac{1}{n^2}\right)$$

$$\boxed{\frac{a_{n+1}}{a_n} = 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right)}$$

$\boxed{\alpha > 1}$

$\alpha > 1$

PRENDO β t.c. $\alpha > \beta > 1$

PRENDO $b_n = \frac{1}{(n-1)^\beta}$ $\quad \sum b_n$ conv.

(def. sn) $\quad \frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ (?)

$$1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right) \leq 1 - \frac{\beta}{n} + o\left(\frac{1}{n}\right) \quad ?$$

$$\frac{\beta - \alpha}{n} + o\left(\frac{1}{n}\right) \leq 0 \quad (?) \quad \boxed{SI} \quad (\beta < \alpha)$$

$$\boxed{\frac{1}{n}\left(\underbrace{(\beta-\alpha)}_{<0} + \underbrace{o(1)}_{0}\right) \leq 0}$$

$$\frac{\sqrt[n]{n!}}{n} = \sqrt[n]{\frac{n!}{n^n}}$$

$$\boxed{\frac{a_{n+1}}{a_n} \to l}$$

$\not\Uparrow \quad \Downarrow$

$\sqrt[n]{a_n} \to l$

$\boxed{T.}$

DATA (a_n) SUCC. A TERM. ST. POS., ALLORA

$$\frac{a_{n+1}}{a_n} \to \ell \in \mathbb{R} \Rightarrow \sqrt[n]{a_n} \to \ell$$

$\boxed{\text{DIM}}$

$\to \forall \varepsilon > 0$ def. in n $\quad \ell - \varepsilon < \sqrt[n]{a_n} < \ell + \varepsilon \quad$ [?]

$\boxed{\text{SI}}$ $\boxed{\frac{a_{n+1}}{a_n} \to \ell}$

PRENDO $n_0 \in \mathbb{N}$ t.c. $\quad \overset{n > n_0}{} \boxed{\frac{a_{n+1}}{a_n} < \ell + \frac{\varepsilon}{2}}$

$\boxed{n \geq n_0}$

$$\sqrt[n]{a_n} = \sqrt[n]{a_{n_0} \frac{a_{n_0+1}}{a_{n_0}} \frac{a_{n_0+2}}{a_{n_0+1}} \cdot \frac{\cdots}{a_{n_0+2}} \cdot \ldots \cdot \frac{a_{n-1}}{\cdots} \cdot \frac{a_n}{a_{n-1}}} \leq$$

$$\leq \sqrt[n]{a_{n_0} \left(\ell + \frac{\varepsilon}{2}\right) \cdots\cdots \left(\ell + \frac{\varepsilon}{2}\right)} =$$

$\forall n > n_0$

$$= \sqrt[n]{a_{n_0} \cdot \left(\ell + \frac{\varepsilon}{2}\right)^{n-n_0}} = \boxed{\underbrace{\sqrt[n]{a_{n_0}}}_{\to 1} \cdot \underbrace{\left(\ell + \frac{\varepsilon}{2}\right)^{1-\frac{n_0}{n}}}_{\ell + \frac{\varepsilon}{2}}} \to \ell + \frac{\varepsilon}{2}$$

DEF IN n $\quad \boxed{\sqrt[n]{a_n} < \left(\ell + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2}}$