LEZ. 13 (11-06-2023)

$\boxed{T.1}$ CR. CONV. ASS.

DATA $\sum a_n$ SE $\sum |a_n|$ CONV. ALLORA $\sum a_n$ CONV.

$\boxed{DIM}$

$$\begin{cases} b_n = \max\{a_n, 0\} \\ c_n = \max\{-a_n, 0\} \end{cases} \qquad c_n = \begin{cases} 0 & \text{SE } a_n \geq 0 \\ -a_n & \text{SE } a_n < 0 \end{cases}$$

$$\left[\sum a_n = \sum (b_n - c_n) \right.$$

$$\left[\sum |a_n| = \sum (b_n + c_n) \right.$$

$$\forall n \in \mathbb{N} \quad \boxed{\begin{aligned} 0 \leq b_n \leq |a_n| \\ 0 \leq c_n \leq |a_n| \end{aligned}}$$

$$\Downarrow$$

$\sum b_n$ $\sum c_n$ CONV.

$$\Downarrow$$

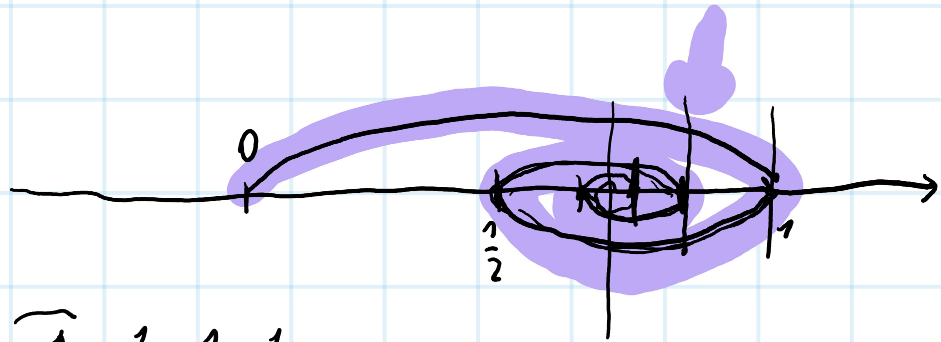
$\sum (b_n - c_n)$ CONV.

$$\Updownarrow$$

$$\sum a_n$$

ES. $\sum \frac{1}{n}$ div.

$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}$ CONV.



$+1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

T.2 CR. LEIBNITZ

DATA $\sum_{n=0}^{+\infty} (-1)^n a_n$ SE $a_n \to 0$ DECR. ALLORA SERIE CONVERGE.

DIM

(?) $S_n \to \ell \in \mathbb{R}$ $\quad S_n = a_0 - a_1 + a_2 - a_3 + \dots + (-1)^n a_n$

I° PASSO S_{2k+1} È CRESC.

$S_{2k+3} \geq S_{2k+1}$?

$$\boxed{S_{2k+3}} = S_{2k+1} + (-1)^{2k+2} a_{2k+2} + (-1)^{2k+3} a_{2k+3} =$$

$$= \boxed{S_{2k+1}} + (a_{2k+2} - a_{2k+3}) \geq \boxed{S_{2k+1}}$$

≥ 0 PERCHÉ $a_{2k+2} \geq a_{2k+3}$

$\boxed{\text{II° PASSO}}$ $\boxed{S_{2k}}$ È DECR. (??)

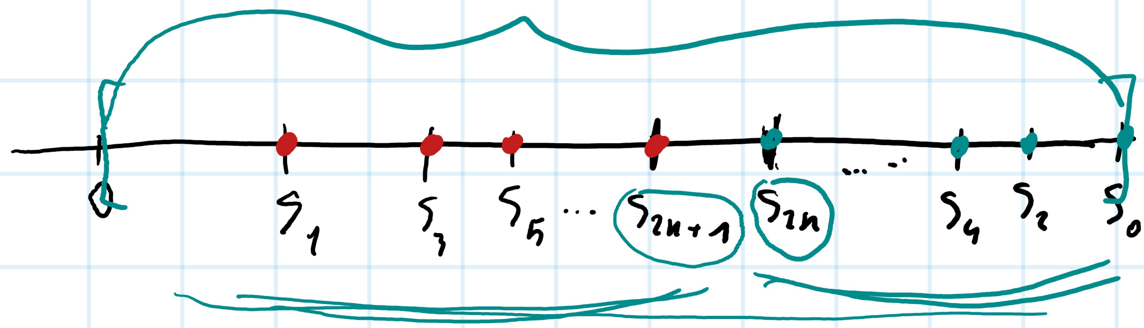
$$S_{2k+2} = S_{2k} + (-1)^{2k+1} a_{2k+1} + (-1)^{2k+2} a_{2k+2} =$$

$$= S_{2k} \underbrace{\boxed{- a_{2k+1} + a_{2k+2}}}_{\le 0} \le S_{2k}$$

$\boxed{\text{III° PASSO}}$

$$\boxed{S_{2k+1}} = S_{2k} + (-1)^{2k+1} a_{2k+1} =$$

$$= S_{2k} - a_{2k+1} \le \boxed{S_{2k}}$$



(S_{2k+1}) E (S_{2n}) SONO IN $[0, a_0]$

QUINDI SONO LIMITATE

QUINDI $\boxed{S_{2n}} \to L \in \mathbb{R}$

$\boxed{S_{2k+1}} \to \ell \in \mathbb{R}$

$$\underbrace{S_{2k+1} - S_{2k}}_{= (-1)^{2k+1} a_{2k+1} = -a_{2k+1} \to 0} \to \boxed{l - L}$$

$$\Downarrow$$

$$l = L$$

$$\Downarrow$$

$$S_n \to l$$

$$\boxed{\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}} \overset{n=m+1}{=} \sum_{m=0}^{+\infty} \frac{(-1)^{m+2}}{m+1} = \sum_{m=0}^{+\infty} (-1)^m \cdot \frac{1}{m+1}$$

a_m

$\boxed{\text{ES. 2}}$

$$\sum_{n=0}^{+\infty} (-1)^n \ln\left(1 + \frac{1}{n+1}\right)$$

NO

$$\sum_{n=0}^{+\infty} (-1)^n \frac{1}{n+1}$$

CONV.

$$\underbrace{a_n = \ln\left(1+\frac{1}{n+1}\right) \to 0 \text{ DECR.}}_{\text{LEIBNIZ}}$$

SI

$$\sum_{n=0}^{+\infty} \ln\left(1+\frac{(-1)^n}{\sqrt{n+1}}\right)$$

NO

$$\approx \boxed{\frac{(-1)^n}{\sqrt{n+1}}}$$

$$\ln\left(1+\frac{(-1)^n}{\sqrt{n+1}}\right) = \underbrace{\cancel{\frac{(-1)^n}{\sqrt{n+1}}}}_{A_n} - \underbrace{\left(\frac{(-1)^n}{\sqrt{n+1}}\right)^2}_{B_n} + \underbrace{O\left(\cancel{\left(\frac{1}{\sqrt{n+1}}\right)^3}\right)}_{C_n}$$

$$\sum \bullet = \sum B_n = \sum_{n=0}^{+\infty} -\frac{1}{n+1} = -\sum_{n=0}^{+\infty} \frac{1}{n+1}$$

$\sum A_n$ CONV. PER L.

$$\left|O\left(\left(\frac{1}{\sqrt{n+1}}\right)^3\right)\right| \le \boxed{\frac{C}{(n+1)^{\frac{3}{2}}}}$$

$\sum \bullet$ CONV.

$$\boxed{\sum O\left(\left(\frac{1}{\sqrt{n+1}}\right)^3\right)} \quad \sum \bullet$$

T.3 CR. (ABEL)

DATA $\sum_{n=0}^{+\infty} a_n b_n$ DOVE

- $a_n \to 0$ DECR.
- b_n T.C. POSTA $B_n = b_0 + \dots + b_n$ SIA (B_n) LIMITATA CIOÈ ESISTE $M>0$ T.C. $|B_n| \le M$

ALLORA ① CONVERGE.

DIM I° PASSO $\boxed{\sum_{n=0}^{+\infty} a_n b_n}$ E $\boxed{\sum_{n=0}^{+\infty} (a_{n+1}-a_n) B_n}$

$a_{n+1} B_n$

$$\mathcal{T}_n = (a_1 - a_0) B_0 + (a_2 - a_1) B_1 + (a_3 - a_2) B_2 + \dots + (a_{n+1} - a_n) B_n$$

$$= -a_0 B_0 - a_1 \underbrace{(B_1 - B_0)}_{b_1} - a_2 \underbrace{(B_2 - B_1)}_{b_2} - a_3 \underbrace{(B_3 - B_2)}_{b_3} \cdots - a_n \underbrace{(B_n - B_{n-1})}_{b_n}$$

$$= \underbrace{-a_0 \underset{b_0}{B_0} + a_{n+1} B_n - (a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n)}_{S_n} =$$

$\mathcal{T}_n$ somma n-esima di $\sum (a_{n+1} - a_n) B_n$

S_n somma n-esima di $\sum a_n b_n$

$$\boxed{\mathcal{T}_n} = \boxed{a_{n+1} B_n} - S_n$$

$$\int_a^b f \cdot g' = [f g]_a^b - \int_a^b f' g$$

II° PASSO $\boxed{\sum (a_{n+1}-a_n) \underset{\downarrow 0}{B_n} \quad \text{CONV.}}$

MOSTRO CONV. DI $\displaystyle\sum_{n=0}^{+\infty} \overbrace{(a_n - a_{n+1})}^{\geq 0} B_n$

$$\sum_{n=0}^{+\infty} \underbrace{\left| (a_n - a_{n+1}) B_n \right|}$$ CONV(?)

$$\boxed{\overbrace{\left| (a_n - a_{n+1}) \cdot B_n \right|}} \leq \overbrace{|a_n - a_{n+1}|} \cdot M = \boxed{(a_n - a_{n+1}) M}$$

$$\boxed{\sum_{n=0}^{+\infty} (a_n - a_{n+1}) \cdot M} = M \boxed{\sum_{n=0}^{+\infty} (a_n - a_{n+1})}$$

$$\boxed{S_n} = (a_0 - \cancel{a_1}) + (\cancel{a_1} - \cancel{a_2}) + (\cancel{a_2} - \cancel{a_3}) + \cancel{\ldots} + (\cancel{a_n} - a_{n+1}) =$$

$$= a_0 - a_{n+1} \to \boxed{a_0}$$

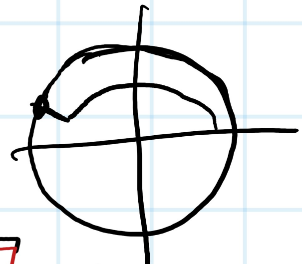
$\Downarrow$

$\sum \left| (a_n - a_{n+1}) \cdot B_n \right|$ CONV. $\Rightarrow \sum (a_n - a_{n+1}) B_n$ CONV.

ES. $\sum_{n=0}^{+\infty} \frac{\cos n}{n+1}$

$S_n = \cos 0 + \cos 1 + \cos 2 + \dots + \cos n$ LIMITATA

$\boxed{e^{in}} = \boxed{\cos n} + i \sin n$



$\mathcal{S}_n = (\cos 0 + \cos 1 + \dots + \cos n) + i(\sin 0 + \sin 1 + \dots + \sin n)$

$= (\cos 0 + i \sin 0) + (\cos 1 + i \sin 1) + \dots + (\cos n + i \sin n) =$

$= e^{0i} + e^{i} + e^{2i} + \dots + e^{ni} =$

$= (e^i)^0 + (e^i)^1 + (e^i)^2 + (e^i)^3 + \dots + (e^i)^n =$

$= \frac{1-e^{(n+1)i}}{1-e^i}$

$1+z+z^2+\dots+z^n = \frac{1-z^{n+1}}{1-z}$

$\left|\frac{1-e^{(n+1)i}}{1-e^i}\right| =$

$= \left|\frac{1}{1-e^i}\right| \cdot \left|1-e^{(n+1)i}\right| \le \left|\frac{1}{1-e^i}\right| \cdot 2$