LEZ. $\boxed{}$ – ES. SERIE

$\boxed{4}$ STUDIARE AL VARIARE DI $\alpha \in \mathbb{R}$ LA CONVERGENZA DI $\sum_{n=1}^{+\infty} a_n$ E $\sum_{n=1}^{+\infty} (-1)^n a_n$

DOVE $a_n = \dfrac{e^n \cdot (n-1)!}{n^{n-\alpha}}$ (NON USARE LA FORMULA DI STIRLING)

$\boxed{5}$ STUDIARE LA CONVERGENZA DI $\sum_{n=1}^{+\infty} a_n$ AL VARIARE DI $\alpha > 0$ NEI SEGUENTI

CASI:

$$a_n = \int_n^{n+1} f(x)\,dx \qquad \text{E} \qquad a_n = \int_n^{n+1} \cos(x) \cdot f(x)\,dx$$

DOVE:

$$f(x) = \frac{1 + x^{2022}}{(1 + x^{\alpha}) \cdot x^{2022}}$$

$\boxed{3}$ $\displaystyle\sum_{n=1}^{+\infty} \frac{n^{2n+3} \cdot A^n}{(2n+1)^n \, (n+1)!}$ AL VARIARE DI $A > 0$

$\boxed{4}$ $\displaystyle\sum_{n=1}^{+\infty} \frac{n - \lfloor \sqrt{n} \rfloor^2}{n^{\alpha}}$ AL VARIARE DI $\alpha > 0$

$\boxed{5}$ SIANO (a_n) E (b_n) A TERMINI POSITIVI E TALI CHE $\sum a_n$ E $\sum b_n$ CONVERGONO.

DIRE QUALI DELLE SEGUENTI SERIE CONVERGONO SICURAMENTE:

a) $\sum_{n=0}^{+\infty} a_n b_n$ b) $\sum_{n=0}^{+\infty} \sqrt{a_n \cdot b_n}$ c) $\sum_{n=0}^{+\infty} \sqrt[3]{a_n b_n}$ d) $\sum_{n=0}^{+\infty} \sqrt{a_n^2 + b_n^2}$ e) $\sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right)$

$\boxed{2}$ POSTO $f(x)=\frac{\sin x}{x}$, SIANO $a_n=\int_n^{n+\frac{1}{n}} f(x)\,dx$ E $b_n=\int_n^{n+1} f(x)\,dx$.

STUDIARE IL CARATTERE DELLE SERIE:

$\boxed{a}$ $\sum_{n=2}^{+\infty} a_n$ $\quad$ $\boxed{b}$ $\sum_{n=2}^{+\infty} b_n$ $\quad$ $\boxed{c}$ $\sum_{n=2}^{+\infty} |b_n|$ ← FACOLTATIVA

$\boxed{2}$ PER OGNI INTERO $n \geq 4$ PONIAMO $a_n=\int_{\ln n}^{\ln(n+1)} x\,dx$. STUDIARE IL CARATTERE DELLE SERIE:

$\boxed{a}$ $\sum_{n=4}^{+\infty} a_n$ $\quad$ $\boxed{b}$ $\sum_{n=4}^{+\infty} (-1)^n a_n$

FACOLTATIVO → $\boxed{c}$ $\sum_{n=4}^{+\infty} (-1)^n b_n$ DOVE $b_n=\int_{\ln n}^{\ln(n+1)} \sqrt{x^2+(-1)^n}\,dx$

$\boxed{2}$ SIA (a_n) DEFINITA PER RICORRENZA NEL MODO SEGUENTE:

$$\begin{cases} a_1 = 1 \\ a_{n+1} = f(a_n) \quad \text{PER OGNI } n \in \mathbb{N}-\{0\} \end{cases}$$

3 PUNTI → $\boxed{a}$ NEL CASO $f(x)=\frac{1}{2}x$ TROVARE ESPLICITAMENTE a_n E STUDIARE IL CARATTERE DI $\sum a_n$.

4 PUNTI → $\boxed{b}$ NEL CASO $f(x)=\frac{1}{2}\arctan x$ STUDIARE IL CARATTERE DI $\sum a_n$.

FACOLTATIVO → $\boxed{c}$ COME $\boxed{b}$ MA CON $f(x)=\arctan x$.

$\boxed{2}$ 7 PUNTI

DIRE, MOTIVANDO LA RISPOSTA, SE È CONVERGENTE LA SERIE:

$$\sum_{n=1}^{+\infty} \frac{\sin\left(\frac{2}{3}\pi n\right)}{\sqrt{n}+\sin\left(\frac{\pi}{4}+n\frac{\pi}{2}\right)}$$

2 PER OGNI INTERO $n \geq 4$ PONIAMO $\boxed{a_n = \int_{\ln n}^{\ln(n+1)} x\, dx}$. STUDIARE IL CARATTERE DELLE SERIE:

$\boxed{\text{a} \quad \sum_{n=4}^{+\infty} a_n \quad \text{(DIVERGE)}}$ $\rightarrow$ $\boxed{\text{b} \quad \sum_{n=4}^{+\infty} (-1)^n a_n}$ CONV.

FACOLTATIVO $\rightarrow$ c $\sum_{n=4}^{+\infty} (-1)^n b_n$ DOVE $b_n = \int_{\ln n}^{\ln(n+1)} \sqrt{x^2 + (-1)^n}\, dx$

$$\boxed{a_n} = \int_{\ln n}^{\ln(n+1)} x\, dx = \left[\frac{x^2}{2}\right]_{\ln n}^{\ln(n+1)} = \frac{1}{2}\left(\ln^2(n+1) - \ln^2 n\right) =$$

$$= \boxed{\frac{1}{2}\left(\ln(n+1) + \ln n\right)} \ln\left(1 + \frac{1}{n}\right) \approx \boxed{\frac{\ln n}{n}}$$

$$\boxed{f(x) = \int_{\ln x}^{\ln(x+1)} t\, dt} \qquad a_n = \int_{\ln n}^{\ln(n+1)} x\, dx$$

$$\boxed{f'(x)} = \underbrace{\ln(x+1) \cdot \frac{1}{x+1} - \ln x \cdot \frac{1}{x}}_{} = H(x+1) - H(x)$$

$$H(x) = \frac{\ln(x)}{x}$$

$$H'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$\boxed{c}$

$$\sum_{n=1}^{+\infty} (-1)^n \left(b_n - a_n\right)$$

$$b_n - a_n = \int_{\ln n}^{\ln(n+1)} \left(\sqrt{x^2+(-1)^n} - x\right) dx$$

$\boxed{n \text{ pari}}$

$$(-1)^n \left(b_n - a_n\right) = \int_{\ln n}^{\ln(n+1)} \sqrt{x^2+1} - x \, dx > 0$$

n dispari

$$(-1)^n \left(b_n - a_n\right) = + \int_{\ln n}^{\ln(n+1)} -\sqrt{x^2-1} + x \, dx > 0$$

$\boxed{n \text{ pari}}$

def. in n

$$(-1)^n \left(b_n - a_n\right) = \int_{\ln n}^{\ln(n+1)} \boxed{\frac{1}{\sqrt{x^2+1} + x}} \, dx \geq \int_{\ln n}^{\ln(n+1)} \boxed{\frac{1}{3x}} \, dx =$$

def per $x \to +\infty$

$$\sqrt{x^2+1} + x < 3x$$

$$= \frac{1}{3}\left(\ln\left(\ln(n+1)\right) - \ln\left(\ln n\right)\right) =$$

$$= \frac{1}{3} \ln\left(\frac{\ln(n+1)}{\ln n}\right) = \frac{1}{3} \ln\left(\frac{\ln n + \overbrace{\ln(n+1) - \ln n}^{\ln\left(1+\frac{1}{n}\right)}}{\ln n}\right) =$$

$$= \frac{1}{3} \ln\left(1 + \boxed{\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}}\right) \sim \frac{1}{3} \cdot \frac{1}{n \ln n}$$

$\boxed{n \text{ dispari}}$

$$(-1)^n(b_n - a_n) = \int_{\ln n}^{\ln(n+1)} \frac{1}{x + \sqrt{x^2-1}}\,dx \geq \boxed{\int_{\ln n}^{\ln(n+1)} \frac{1}{3x}\,dx} =$$

$$\ldots\ldots \approx \frac{1}{3}\,\frac{1}{n \ln n}$$

$\sum \frac{1}{n \ln n}$ diverge

$$\underbrace{(-1)^n(b_n - a_n)}_{\boxed{\text{DIV.}}} \geq \underbrace{\boxed{\mathcal{U}}}_{\boxed{\text{DIV}}} \approx \frac{1}{3}\,\underbrace{\frac{1}{n \ln n}}_{\text{DIV.}}$$

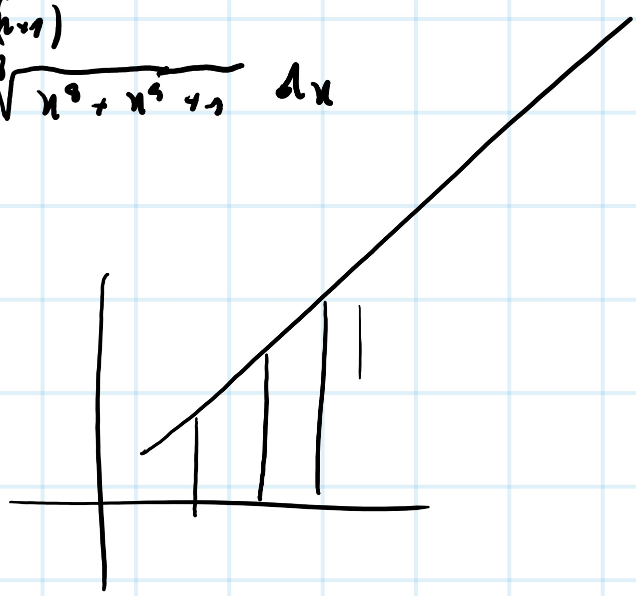
$\Leftarrow$

$\sum_n \int_{\ln(n)}^{\ln(n+1)} x\,dx$ diverge

$$S_n = \int_{\ln 2}^{\ln(n+1)} x\,dx =$$

$$\lim_{n\to+\infty} S_n = \lim_{b\to+\infty} \int_{\ln 2}^{\ln(b)} x \, dx =$$

$$\int_{\ln n}^{\ln(n+1)} \sqrt[4]{x^8 + x^6 + 1} \, dx$$



2 SIA (a_n) DEFINITA PER RICORRENZA NEL MODO SEGUENTE:

$$\begin{cases} a_1 = 1 \\ a_{n+1} = f(a_n) \quad \text{PER OGNI } n \in \mathbb{N} - \{0\} \end{cases}$$

(3 PUNTI) → **a** NEL CASO $f(x) = \frac{1}{2}x$ TROVARE ESPLICITAMENTE a_n E STUDIARE IL CARATTERE DI $\sum a_n$.

(4 PUNTI) → **b** NEL CASO $f(x) = \frac{1}{2}\arctan x$ STUDIARE IL CARATTERE DI $\sum a_n$.

(FACOLTATIVO) → **c** COME **b** MA CON $f(x) = \arctan x$.

$\boxed{a}$

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \frac{1}{2} a_n \end{cases}$$

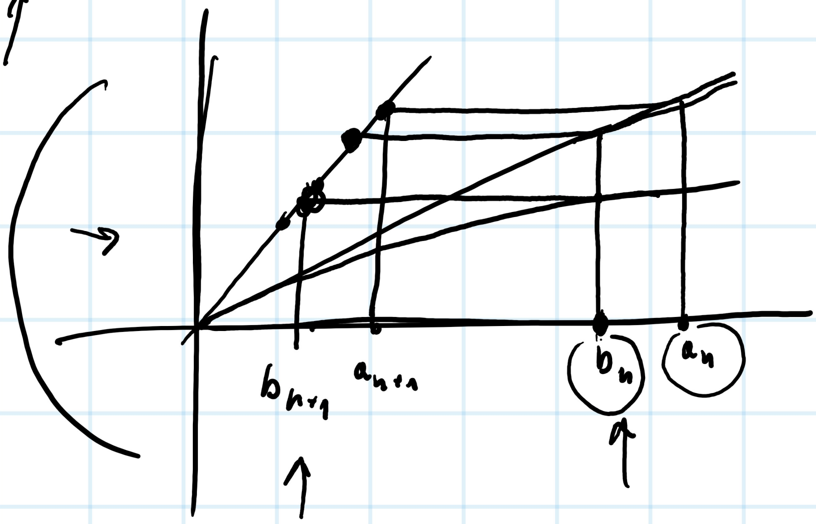
$$a_n = \frac{1}{2^{n-1}} \qquad \sum a_n = \sum \frac{1}{2^{n-1}} = \text{converge.}$$

$f(x) = \frac{1}{2}\arctan x$

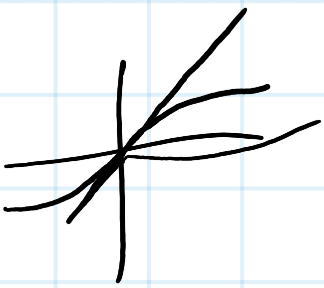
$\boxed{b}$

$$\begin{cases} b_1 = 1 \\ \boxed{b_{n+1} = \frac{1}{2}\arctan b_n} \end{cases}$$

$\sum a_n$



$$\boxed{\begin{aligned} & x \geq 0 \\ & \arctan x \leq x \end{aligned}}$$



$$\boxed{\frac{b_{n+1}}{b_n}} = \frac{\frac{1}{2}\arctan b_n}{b_n} \leq \frac{\frac{1}{2} b_n}{b_n} = \boxed{\frac{1}{2}}$$

$\sum b_n$ converge

$\sum c_n$

$$\boxed{\begin{cases} c_1 = 1 \\ c_{n+1} = \arctan c_n \end{cases}}$$

$\boxed{c_n} \geqslant \boxed{\frac{1}{n}}$ $\quad \forall n \in \mathbb{N} \setminus \{0\}$

$$\boxed{\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

$\boxed{\arctan x} = \boxed{x - \frac{x^3}{3} + \boxed{\cdots}} \geqslant$ $\quad \cdots \geq 0$

$\geqslant x - \frac{x^3}{3}$

$c_n \geq \frac{1}{n}$

$$c_{n+1} = \arctan c_n \geqslant \arctan \frac{1}{n} \geqslant \left(\frac{1}{n} - \frac{1}{2n^3}\right) \geqslant \frac{1}{n+1}$$

(S1)

$\forall n \quad \boxed{\boxed{c_n} \geqslant \boxed{\frac{1}{n}}}$

$$\boxed{\frac{1}{n} - \frac{1}{3n^3} \geqslant \frac{1}{n+1}} \;?$$

$$\frac{1}{n} - \frac{1}{n+1} \geqslant \frac{1}{3n^3} \;?$$

$$\boxed{\frac{1}{n} - \frac{1}{2n^3} \geqslant \frac{1}{n+1}}$$

S1

$$\frac{1}{n(n+1)} \geqslant \frac{1}{3n^3} \;(?)$$

$$\frac{1}{n+1} \geqslant \frac{1}{3n^2} \;(?)$$

$$n+1 \leq 3n^2 \;(?)$$

2

$$\arctan \frac{1}{n} \overset{?}{\geq} \frac{1}{n} - \frac{1}{2n^3}$$

$$\arctan \frac{1}{n} = \frac{1}{n} - \frac{1}{3n^3} + o\left(\frac{1}{n^3}\right) =$$

$$= \frac{1}{n} - \frac{1}{2n^3} + \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right) =$$

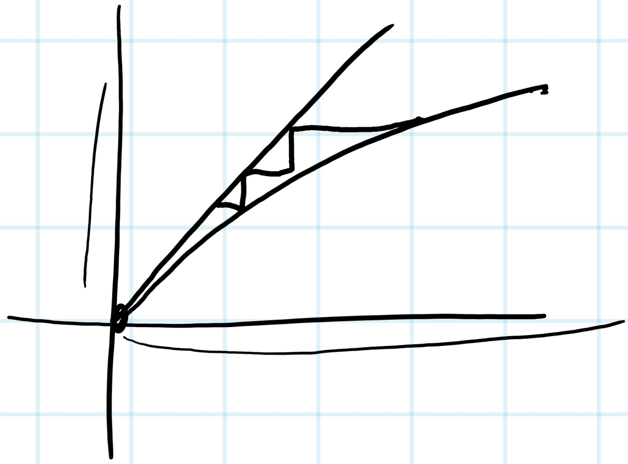
$$= \frac{1}{n} - \frac{1}{2n^3} + \frac{1}{n^3}\left(\frac{1}{6} + o(1)\right)$$

$$\frac{c_{n+1}}{c_n} = \frac{\operatorname{arctan} c_n}{c_n} = \frac{c_n - \frac{1}{3}(c_n)^3 + O(c_n^4)}{c_n} = 1 - \frac{1}{3}\cdot c_n^2 + O(\ldots$$

$o\left(\frac{1}{n}\right)$

$$\frac{c_{n+1}}{c_n} \approx 1 - \frac{\alpha}{n}$$

$1 - o\left(\frac{1}{n^2}\right)$ DIV.



$$\frac{a_{n+1}}{a_n} \approx 1 - o\left(\frac{1}{n^2}\right) \qquad \text{DIV.}$$

$\forall \alpha > 0$

$$1 - \underbrace{o\left(\frac{1}{n^2}\right)}_{o\left(\frac{1}{n^2}\right)} \geq \boxed{1 - \frac{\alpha}{n}}$$

$$o\left(\frac{1}{n^2}\right) \leq \frac{\alpha}{n} \qquad \forall \alpha > 0$$

SI

2 POSTO $f(x)=\boxed{\frac{\sin x}{x}}$, SIANO $a_n=\int_n^{n+\frac{1}{n}} f(x)\,dx$ E $b_n=\int_n^{n+1} f(x)\,dx$.

STUDIARE IL CARATTERE DELLE SERIE:

a $\sum_{n=2}^{+\infty} a_n$ **b** $\sum_{n=2}^{+\infty} b_n$ **c** $\boxed{\sum_{n=2}^{+\infty} |b_n|}$ ← FACOLTATIVA

$$a_n = \int_n^{n+1} f(x)\,dx$$

$$S_n = \sum_{k=1}^{n} a_k = \int_1^2 f + \int_2^3 f + \int_3^4 \cdots + \int_{n}^{n+1} =$$

$$= \boxed{\int_1^{n+1} f(x)\,dx}$$

$$\lim_{n\to+\infty} S_n = \; = \int_1^{+\infty} f(x)\,dx$$

$$\int_1^{+\infty} \frac{\sin x}{x}\,dx$$

$$|b_n| = \left|\int_n^{n+\frac{1}{n}} \frac{\sin x}{x}\,dx\right| \leq \int_n^{n+\frac{1}{n}} \left|\frac{\sin x}{x}\right| dx \leq \int_n^{n+\frac{1}{n}} \frac{1}{n}\,dx$$

$\leq \frac{1}{n}$

$$\sum |b_n|$$

$$\leq \frac{1}{n}\int_n^{n+\frac{1}{n}} 1\,dx = \frac{1}{n}\cdot\frac{1}{n} = \boxed{\frac{1}{n^2}}$$

$$\boxed{|b_n| \leq \frac{1}{n^2}}$$