LEZ. 15 - ULTIMI EXE SERIE + INIZIO $\mathbb{C}$ (17/04/2023)

4 STUDIARE AL VARIARE DI $\alpha \in \mathbb{R}$ LA CONVERGENZA DI $\sum_{n=1}^{+\infty} a_n$ E $\sum_{n=1}^{+\infty} (-1)^n a_n$

DOVE $a_n = \dfrac{e^n \cdot (n-1)!}{n^{n-\alpha}}$ (NON USARE LA FORMULA DI STIRLING)

5 STUDIARE LA CONVERGENZA DI $\sum_{n=1}^{+\infty} a_n$ AL VARIARE DI $\alpha > 0$ NEI SEGUENTI CASI:

$$a_n = \int_n^{n+1} f(x)\,dx \quad \text{E} \quad a_n = \int_n^{n+1} \cos(x) \cdot f(x)\,dx$$

DOVE:

$$f(x) = \frac{1 + x^{2022}}{(1 + x^{\alpha}) \cdot x^{2022}}$$

3 $\displaystyle\sum_{n=1}^{+\infty} \frac{n^{2n+1} \cdot A^n}{(2n+1)^n (n+1)!}$ AL VARIARE DI $A > 0$

4 $\displaystyle\sum_{n=1}^{+\infty} \frac{n - \lfloor \sqrt{n} \rfloor^2}{n^{\alpha}}$ AL VARIARE DI $\alpha > 0$

5 SIANO (a_n) E (b_n) A TERMINI POSITIVI E TALI CHE $\sum a_n$ E $\sum b_n$ CONVERGONO.

DIRE QUALI DELLE SEGUENTI SERIE CONVERGONO SICURAMENTE:

(a) $\sum_{n=0}^{+\infty} a_n b_n$ (b) $\sum_{n=0}^{+\infty} \sqrt{a_n \cdot b_n}$ (c) $\sum_{n=0}^{+\infty} \sqrt[3]{a_n b_n}$ (d) $\sum_{n=0}^{+\infty} \sqrt{a_n^2 + b_n^2}$ (e) $\sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right)$

NUMERI COMPLESSI

1) 2 DEFINIZIONI
2) FORMA TRIGON.
3) RADICI n-ESIME
4) e^z

(1) $\sum a_n \cdot b_n$ CONV.?

$\sum a_n$ CONV

$\sum b_n$ CONV. $\Rightarrow b_n \to 0 \Rightarrow$ DEF. IN n $\overbrace{b_n \leq 1}$

$\Downarrow$

DEF. IN n $a_n b_n \leq a_n \cdot 1 = a_n$

$$\boxed{0 \leq a_n b_n \leq a_n \quad e \quad \sum a_n \text{ CONV.}}$$

$\Downarrow$ C.CONF

$\sum a_n b_n$ CONVERGE

$\boxed{\text{SE NON SONO} \geq 0 \text{ NON VALE}}$

$\sum \frac{(-1)^n}{\sqrt{n}}$ CONV. PER. LEIB.

$a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$

$\sum a_n b_n = \sum \frac{((-1)^n)^2}{n} = \sum \frac{1}{n}$

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(2) $\sum \sqrt{a_n \cdot b_n}$ CONV.?

$\to 2\sqrt{a_n \cdot b_n} \leq a_n + b_n$ ~~(?)~~ SI

$\to 4 a_n b_n \leq (a_n + b_n)^2$ (?)

$\to 4 a_n b_n \leq a_n^2 + b_n^2 + 2 a_n b_n$ (?)

$\to 0 \leq a_n^2 + b_n^2 - 2 a_n b_n$ (?)

$$\rightarrow \boxed{0 \leq (a_n - b_n)^2} \quad (?)$$

$$0 \leq \boxed{\sqrt{a_n b_n}} \leq \frac{a_n + b_n}{2}$$

CONV.

$$\left.\begin{array}{l}\sum a_n \text{ CONV.}\\ \sum b_n \text{ CONV.}\end{array}\right\} \Rightarrow \sum \frac{a_n + b_n}{2} \text{ CONV.}$$

$\sum \sqrt{a_n b_n}$ CONV. (CR. CONF.)

$$\sum \sqrt[3]{a_n b_n}$$

$$b_n = a_n = \frac{1}{n\sqrt{n}} \qquad \sum \frac{1}{n\sqrt{n}} \text{ CONV.}$$

$$\sqrt[3]{a_n b_n} = \sqrt[3]{\left(\frac{1}{n\sqrt{n}}\right)^2} = \frac{1}{n}$$

$$\sum \sqrt{a_n^2 + b_n^2}$$

$$0 \leq \sqrt{a_n^2 + b_n^2} \leq \boxed{a_n + b_n} \quad (?)$$

$$\underline{a_n^2 + b_n^2 \leq a_n^2 + b_n^2 + 2 a_n b_n} \quad (?)$$

$$\sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) = \underline{\sum A_n}$$

$$A_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0$$

$$\to \underbrace{(a_0 b_0)}_{A_0} + \underbrace{(a_0 b_1 + a_1 b_0)}_{A_1} + \underbrace{(a_0 b_2 + a_1 b_1 + a_2 b_0)}_{A_2} + \dots$$

$$(a_0 + a_1 + \dots + a_n) \cdot (b_0 + b_1 + \dots + b_n)$$

$$(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)(b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \dots +$$

$$+ (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0) x^n +$$

$\to s_n = a_0 + a_1 + \dots + a_n \qquad s_n \to \alpha \in \mathbb{R}$

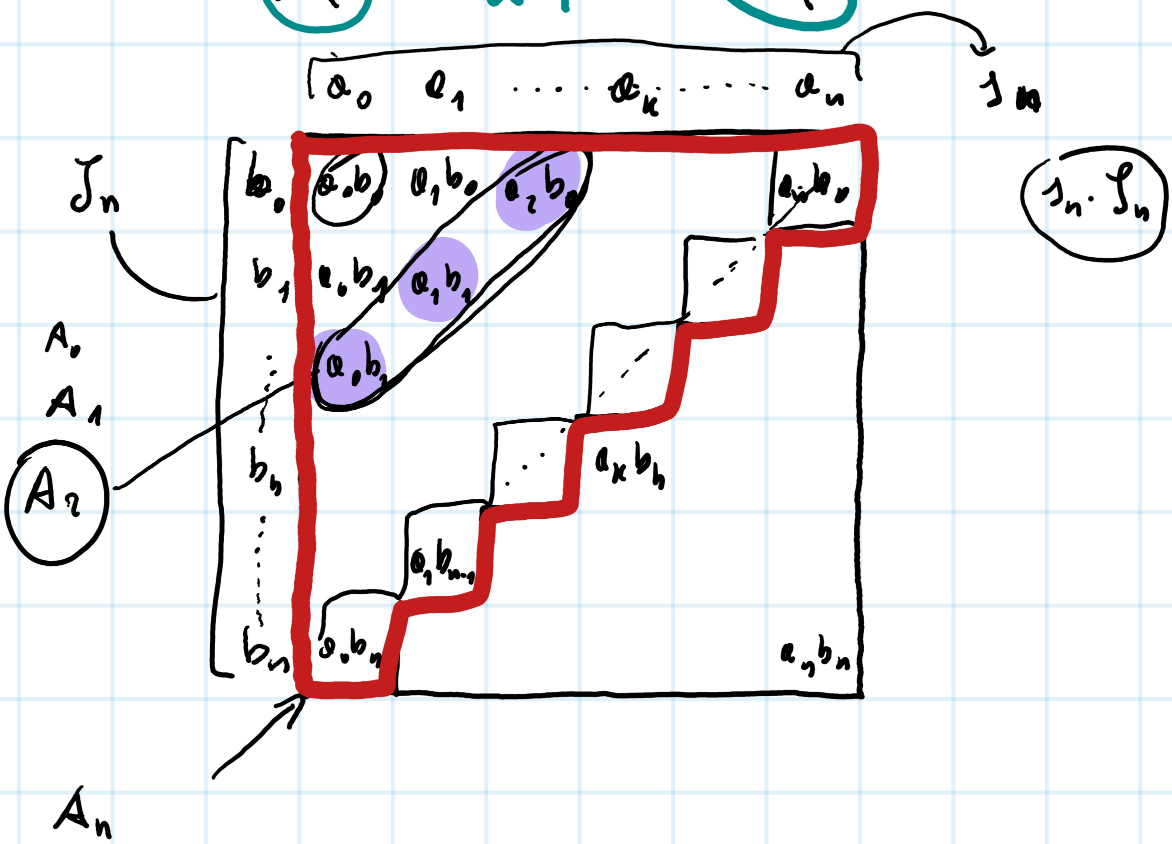
$\to \sigma_n = b_0 + b_1 + \dots + b_n \qquad \sigma_n \to \beta \in \mathbb{R}$

$$S_n = A_0 + A_1 + \dots + A_n = \qquad S_n \overset{?}{\to} \alpha \cdot \beta$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots + (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0)$$

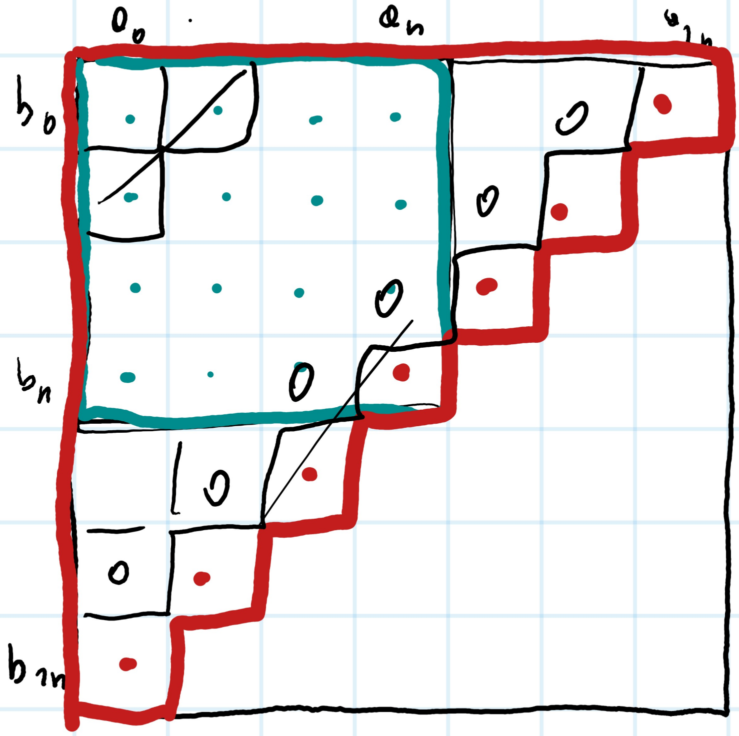
$\boxed{S_{2k} \to \alpha \cdot \beta}$?

$$\underbrace{J_k \cdot \mathcal{J}_k}_{\alpha\cdot\beta} \le \underbrace{S_{2k}}_{\alpha\cdot\beta} \le \underbrace{J_{2k} \cdot \mathcal{J}_{2k}}_{\alpha\cdot\beta}$$



$$\square = A_0 + A_1 + \cdots + A_n = S_n$$

$$\blacksquare = (a_0 + a_1 + \cdots + a_n)\cdot(b_0 + b_1 + \cdots + b_n) = s_n \cdot \mathcal{S}_n$$



$$\square = \overbrace{(a_0 b_0)}^{A_0} + \overbrace{(a_0 b_1 + a_1 b_0)}^{A_1} + \cdots\cdots + \overbrace{\underline{(a_0 b_{2n} + a_1 b_{2n-1} + \cdots + a_{2n} b_0)}}^{A_{2n}}$$

$$= S_{2n}$$

$$\boxed{\overbrace{s_n \cdot \mathcal{S}_n}^{\blacksquare} \le \overbrace{S_{2n}}^{\blacksquare}}$$

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DOVE $a_n = \dfrac{e^n \cdot (n-1)!}{n^{n-\alpha}}$ (NON USARE LA FORMULA DI STIRLING)

5 STUDIARE LA CONVERGENZA DI $\sum_{n=1}^{+\infty} a_n$ AL VARIARE DI $\alpha > 0$ NEI SEGUENTI

CASI:

$$a_n = \int_n^{n+1} f(x)\,dx \quad \text{E} \quad \boxed{a_n = \int_n^{n+1} \cos[x] \cdot f(x)\,dx} = \cos n \cdot \int_n^{n+1} f(x)\,dx$$

DOVE:

$$\boxed{f(x) = \frac{1 + x^{2022}}{(1+x^\alpha) \cdot x^{2022}}} \overset{x \to +\infty}{\approx} \frac{1}{x^\alpha} \quad \boxed{\alpha > 1}$$

$$a_n = \int_n^{n+1} \frac{1+x^{2022}}{(1+x^\alpha)\,x^{2022}}\,dx$$

$$\sum a_n$$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} (a_1 + a_2 + \dots + a_n) = \lim_{n\to+\infty} \int_1^{n+1} f(x)\,dx = \int_1^{+\infty} f(x)\,dx$$

$$\underbrace{\int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \dots + \int_n^{n+1} f(x)\,dx}$$

$$\boxed{\int_1^{n+1} f(x)\,dx}$$

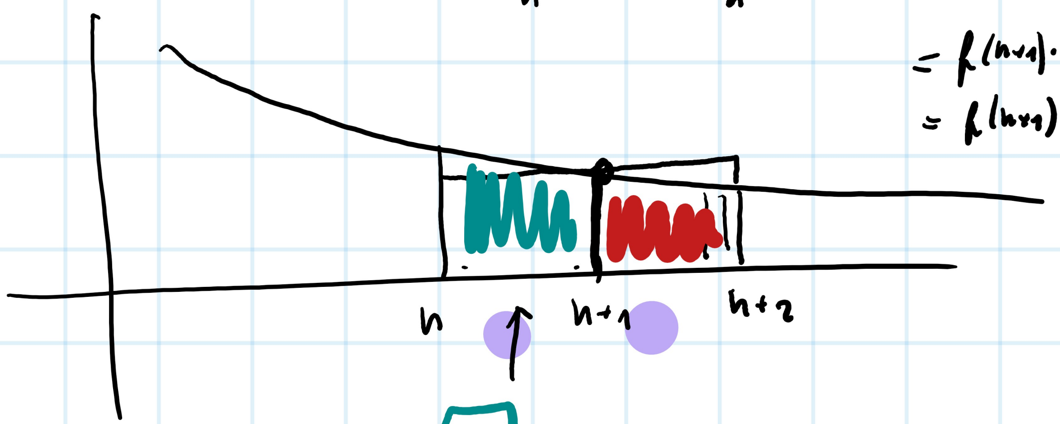
$\sum b_n a_n$

$\searrow 0$

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \cos n \cdot \boxed{\int_n^{n+1} \frac{1+x^{2022}}{(1+x^{\alpha})\,x^{2022}}\,dx}$$

$B_n = \sum_{k=1}^{n} \cos k$ $\qquad (B_n)$ è limitata

$$a_n = \int_n^{n+1} f(x)\,dx > \int_n^{n+1} \boxed{f(n+1)}\,dx = f(n+1)\cdot\int_n^{n+1} 1\,dx = f(n+1)\cdot 1 = f(n+1)$$



$$a_n \geq f(n+1) \geq a_{n+1}$$

$$f(x) = \frac{1+x^{2022}}{x^{2022}} \cdot \frac{1}{1+x^{\alpha}} =$$

$$= \left(1 + \frac{1}{x^{2022}}\right) \cdot \frac{1}{1+x^{\alpha}}$$

$$\boxed{\mathbb{C}}$$

$$\mathbb{R}$$

$$a + ib$$

$$\mathbb{R}^2$$

$$(a,b) + (\alpha,\beta) = (a+\alpha,\ b+\beta)$$

$$(a,b) \cdot (\alpha,\beta) = (a\alpha - b\beta,\ a\beta + b\alpha)$$

$$\boxed{(1,0)}$$

$$(a,b)\cdot(1,0) = (a,b)$$

$$\underbrace{(a,b)}\cdot\overbrace{\underbrace{\left(\frac{a}{a^2+b^2},\ \frac{-b}{a^2+b^2}\right)}} = \left(a\cdot\frac{a}{a^2+b^2} - b\cdot\frac{-b}{a^2+b^2},\ \cancel{a\cdot\frac{-b}{a^2+b^2}} + \cancel{b\cdot\frac{a}{a^2+b^2}}\right) =$$

$$= \left(\frac{a^2+b^2}{a^2+b^2},\ 0\right) = (1,0)$$

$$(a,b)\cdot(\alpha,\beta) = \underbrace{(a\alpha - b\beta,\ a\beta + b\alpha)}$$

$$(a+ib)\cdot(\alpha+i\beta) = a\alpha + ib\alpha + ia\beta + \underbrace{ib\cdot i\beta} =$$

$$= \underbrace{(a\alpha - b\beta) + (b\alpha + a\beta)}\, i$$

$\boxed{\text{DEF}}$ DATO $z = a+ib \in \mathbb{C}$ DEFINIAMO

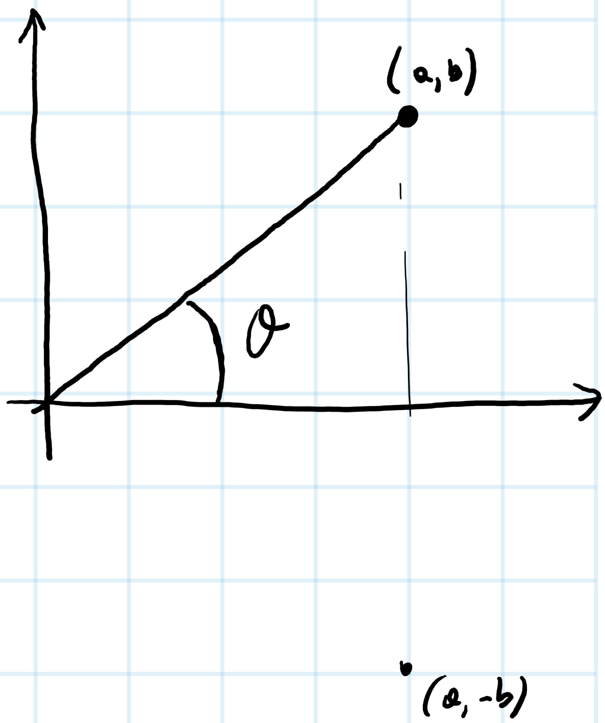
$$|z| = \sqrt{a^2+b^2}$$

quell'unico

$$\arg(z) = \theta \in [0, 2\pi) \text{ t.c.}$$

$$\cos\theta = \frac{a}{|z|}$$

$$\sin\theta = \frac{b}{|z|}$$



$$z = a+bi$$

$$\bar{z} = \boxed{a-bi}$$

$\boxed{\text{T.}}$ DATI $z_1 = a_1 + i b_1$ $\quad z_2 = a_2 + i b_2$ SIANO

$\rho_1 = |z_1|$, $\rho_2 = |z_2|$ $\quad \theta_1 = \arg(z_1)$ $\quad \theta_2 = \arg(z_2)$.

ALLORA DETTO $z = z_1 \cdot z_2$ SI HA

$$\left\{ \begin{array}{l} |z| = \rho_1 \cdot \rho_2 \\ \arg(z) - (\theta_1 + \theta_2) \text{ È MULT. DI } 2\pi \end{array} \right.$$

$\boxed{\text{DIM}}$

$a_1 = \rho_1 \cos\theta_1 \qquad a_2 = \rho_2 \cos\theta_2$

$b_1 = \rho_1 \sin\theta_1 \qquad b_2 = \rho_2 \sin\theta_2$

$$z = z_1 \cdot z_2 = (\rho_1 \cos\theta_1 + i \rho_1 \sin\theta_1) \cdot (\rho_2 \cos\theta_2 + i \rho_2 \sin\theta_2) =$$

$$= \rho_1 \cdot \rho_2 (\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i \sin\theta_2) =$$

$$= \rho_1 \cdot \rho_2 \left(\underbrace{(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)} + (\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1) i \right)$$

$$= \rho_1 \cdot \rho_2 \cdot \left(\underbrace{\cos(\theta_1 + \theta_2)} + \underbrace{\sin(\theta_1 + \theta_2)} i \right)$$

$\boxed{\text{DEF.}}$ DATO $z = a + ib \in \mathbb{C}$ DEFINIAMO

$$e^z = e^{a+ib} = e^a \cdot (\cos b + i \sin b)$$

$$e^{z_1} \cdot e^{z_2} \overset{?}{=} e^{z_1+z_2}$$

$z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$

$$e^{z_1} \cdot e^{z_2} = e^{a_1+ib_1} \cdot e^{a_2+ib_2} =$$

$$= e^{a_1} \cdot \underbrace{(\cos b_1 + i \sin b_1)} \cdot e^{a_2} \cdot \underbrace{(\cos b_2 + i \sin b_2)} =$$

$\boxed{\text{STESSI CONTI}}$

$$= [\cdots] = e^{a_1+a_2} \cdot (\cos(b_1+b_2) + i \sin(b_1+b_2)) =$$

$$= e^{(a_1+a_2) + i(b_1+b_2)}$$

$$\mathbb{C} \to \mathbb{C}$$
$$z \mapsto e^z$$