AM2 - LEZ. EQD01 - 02 - EQUAZIONI DIFFERENZIALI (TEORIA GENERALE)

(22/05/2023)

(23/05/2023)

0) ESEMPIO INTRODUTTIVO $y' = 2y$

1) SOLUZ. a) DEF (PAG 53)
   b) EX 1: $y' = 2xy$ (PAG. 1)
   c) EX 2: $y' = y^2$

2) TEO. ESISTENZA E UNICITÀ (PAG. 55)

3) COROLLARIO: (I_1, y_1) E (I_2, y_2) SOL P.C. $\Rightarrow$ $y_1 \equiv y_2$ SU $I_1 \cap I_2$ (PAG. 55)

4) DEF. PROLUNGAMENTO E SOL. MASSIMALE (PAG 57) + $\boxed{\text{ESEMPI}}$

5) TEO. ESIST. SOL. MAX. (PAG. 57)

6) EQUADIFF. VAR. SEP.
   a) DEF. (PAG. 56)
   b) EX 1: $y' = 2xy(y-1)$ $y(0) = 0, 1, \frac{1}{2}, 2, -1$ (PAG. 10)

LEZ. 1

   c) EX 2: $y' = e^x \cos^2(y)$ $y(0) = \frac{\pi}{2}, \frac{\pi}{4}, \frac{5}{4}\pi$ (PAG. 3)

LEZ 2

   d) EX 3: (CATTIVO) $y' = \sqrt[3]{y}$ (PAG. 7)

7) TEO. PROLUNGAMENTO FUORI DAI COMPATTI (PAG. 58)

8) EX. DI APPLICAZIONE: $y' = \arctan(y(y-1))$ $y(0) = \frac{1}{2}$ (PAG. 13)

9) SOPRASOLUZIONI E SOTTOSOLUZIONI (DEF E TEO) (PAG. 65 - 66 - 67)

10) T. CONFRONTO (PAG. 67)

11) ESEMPIO DI APPLICAZIONE: $\begin{cases} y' = y^2 - x^2 \\ y(0) = \alpha \end{cases}$ (PAG. 67)

$$\boxed{y' = 2y}$$

$$y'_{(x)} = F(x, y(x))$$

$$\boxed{y'(x) = 2y(x)}$$

$$\boxed{y' = F(x,y)}$$

$$\boxed{y(x) = e^{2x}}$$

$$\boxed{y(x) = Ke^{2x}}$$

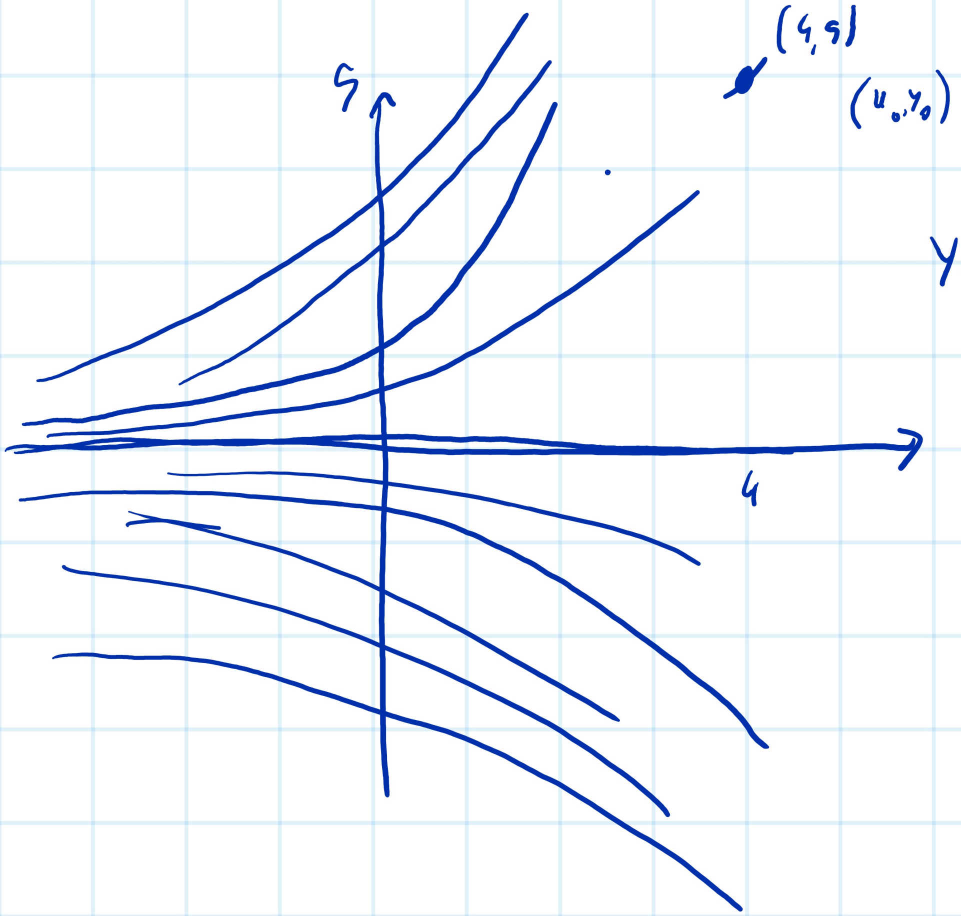
$$(ke^{2x})' = \underbrace{k2e^{2x}}$$

$$2 \cdot ke^{2x} =$$

$$y(x) = Ke^{2x}$$

$$5 = ke^{8}$$

$$k = \frac{5}{e^{8}}$$



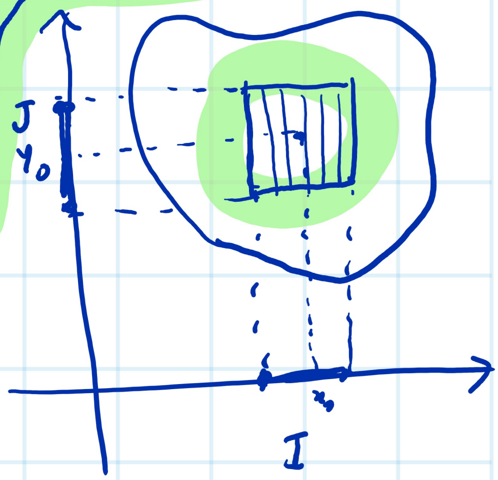
$$y(x) = \frac{5}{e^{8}} \cdot e^{2x} = \boxed{5e^{2x-8}}$$

$\boxed{\text{DEF}}$ DATO $\Omega \subset \mathbb{R}^2$ aperto, $(x_0,y_0) \in \Omega$, $F : \Omega \to \mathbb{R}$ t.c. F è continua,

$(\mathcal{L})$ $\exists\, I$ intorno di x_0 e $\exists\, J$ intorno y_0 t.c. $I \times J \subset \Omega$ e

$\exists\, L > 0$ t.c. $\forall x \in I$ $\forall y_1, y_2 \in J$

$$|F(x,y_1) - F(x,y_2)| \leq L\,|y_1 - y_2|$$



E DATA $((\alpha,\beta),\ y(x))$ con $y(x) \in C^1((\alpha,\beta))$

DIRÒ CHE $((\alpha,\beta), y(x))$ È SOLUZIONE DI

$$y' = F(x,y)$$

SE $\forall x \in (\alpha,\beta)$ $\underline{(x, y(x)) \in \Omega}$ E $y(x)$ SODDISFI

$$\underline{y'(x) = F(x, y(x))}$$

INOLTRE DIREMO CHE È SOL. DEL P. CAUCHY

$$\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

SE VALE

$\boxed{y' = y^2}$

$$\boxed{y(x) = -\frac{1}{x+c}} \qquad c \in \mathbb{R}$$

$y' = \sqrt{xy}$

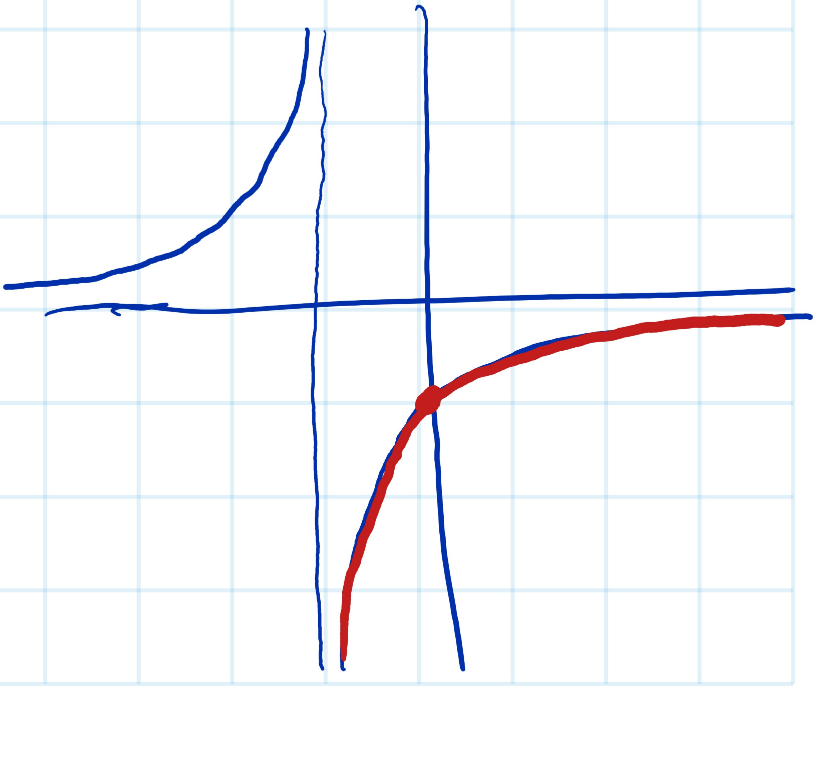
$$y'(x) = -\cdot\left(-\frac{1}{(x+c)^2}\right)\cdot 1 = \frac{1}{(x+c)^2}$$

$$y(x) = -\frac{1}{x+1}$$

$$\to \left(\underline{(-\infty, -1)},\ \underline{y(x) = -\frac{1}{x+1}}\right) \leftarrow$$

$$\to \boxed{\left((-1, +\infty),\ y = -\frac{1}{x+1}\right)} \leftarrow \qquad y(x) = -\frac{1}{x+1}$$

$$\begin{cases} \underline{y' = y^2} \\ \underline{y(0) = -1} \end{cases}$$



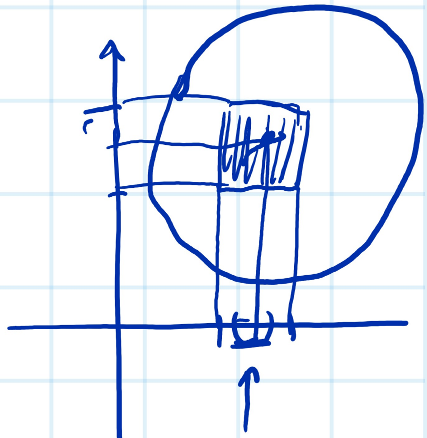
$\boxed{\text{T. E\&U locale}}$ DATI $\Omega \subset \mathbb{R}^2$ aperto, $(x_0,y_0) \in \Omega$, $F: \Omega \to \mathbb{R}$ continua

e t.c. soddisfa (L)

(L) $\exists\, I$ intorno di x_0 e $\exists\, J$ intorno di y_0 t.c. $I \times J \subset \Omega$ e

$\exists\, L > 0$ t.c. $\forall x \in I\ \forall y_1, y_2 \in J$

$$|F(x,y_1) - F(x,y_2)| \leq L\,|y_1 - y_2|$$

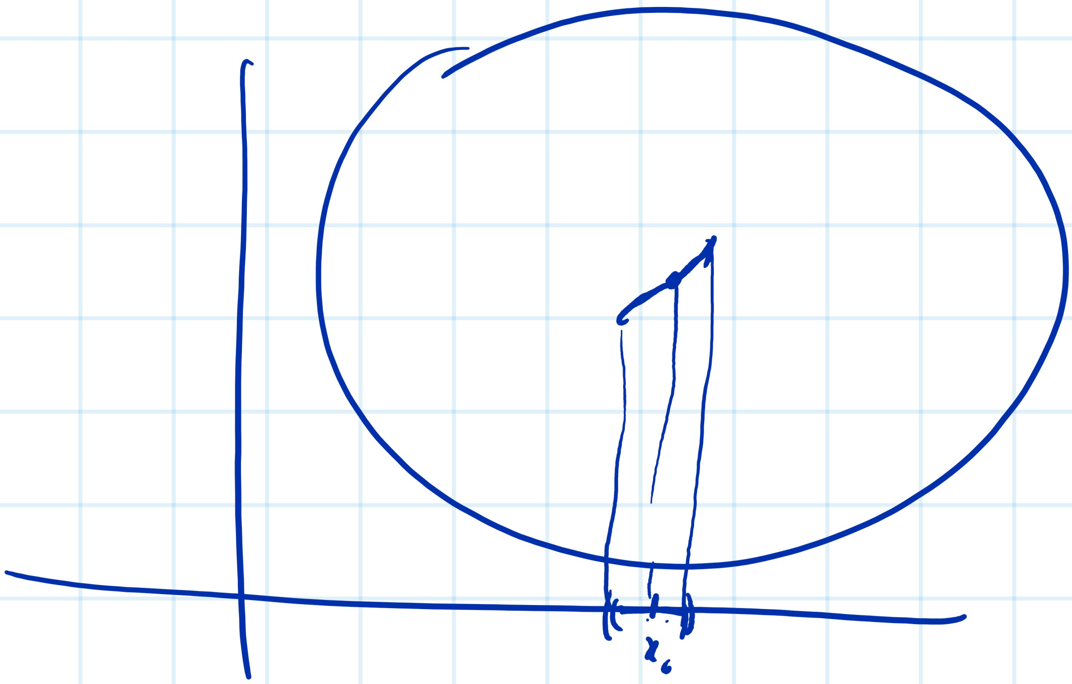


PRESO

$$(*) \quad \boxed{\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}}$$

$\exists\, \delta > 0$ $\exists!\, y(x) \in C^1((x_0-\delta, x_0+\delta))$ t.c.

$((x_0-\delta, x_0+\delta), y(x))$ È SOL. DI $(*)$

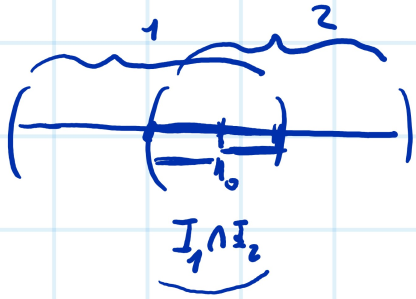


$\boxed{\text{TEO}}$ DATO $\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$ CON IPOTESI STANDARD.

CIOÈ $(\mathcal{L})$ VALE NON SOLO IN (x_0,y_0) MA IN OGNI $(\bar{x},\bar{y}) \in \Omega$

SIANO $(I_1, y_1(x))$ E $(I_2, y_2(x))$ 2 SUE SOLUZIONI ALLORA $\forall x \in I_1 \cap I_2$ SI HA $y_1(x) = y_2(x)$

$\boxed{\text{DIM.}}$



SIA $A^+ = \{ x \geq x_0 \mid x \in I_1 \cap I_2 \text{ E } y_1(x) \neq y_2(x) \}$

SE $A^+ \neq \emptyset$ SIA $\bar{x} = \inf A^+$

$\bar{x} \geq x_0$

MOSTRIAMO CHE $y_1(\bar{x}) = y_2(\bar{x})$

$\bar{x} = x_0$ (ovvio)

$\bar{x} > x_0$ $\boxed{y_1(x) = y_2(x)}$ $\forall x \in [x_0, \bar{x})$

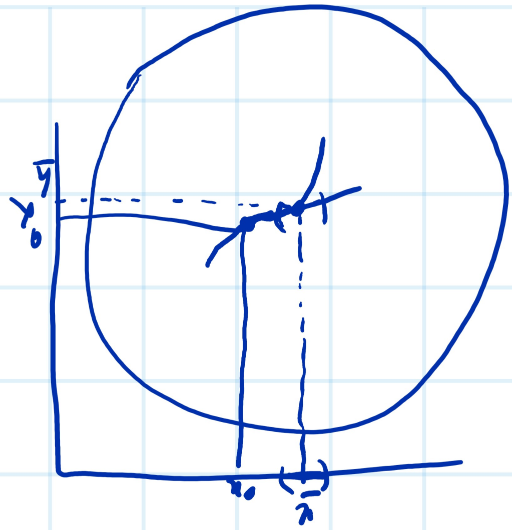
quindi $d(x) = y_1(x) - y_2(x) = 0$ su $[x_0, \bar{x})$

ed è continua anche in $\bar{x}$ quindi $d(\bar{x}) = 0$

MOSTRIAMO CHE $\exists \delta > 0$ t.c. $y_1(x) = y_2(x)$ su $(\bar{x}, \bar{x}+\delta)$

SIA $\bar{y} = y_1(\bar{x}) = y_2(\bar{x})$ E CONSIDERO

$$(*)\quad \begin{cases} y' = F(x,y) \\ y(\bar{x}) = \bar{y} \end{cases}$$



$\exists\ \underbrace{(\bar{x}-\delta,\ \bar{x}+\delta)}_{I}$

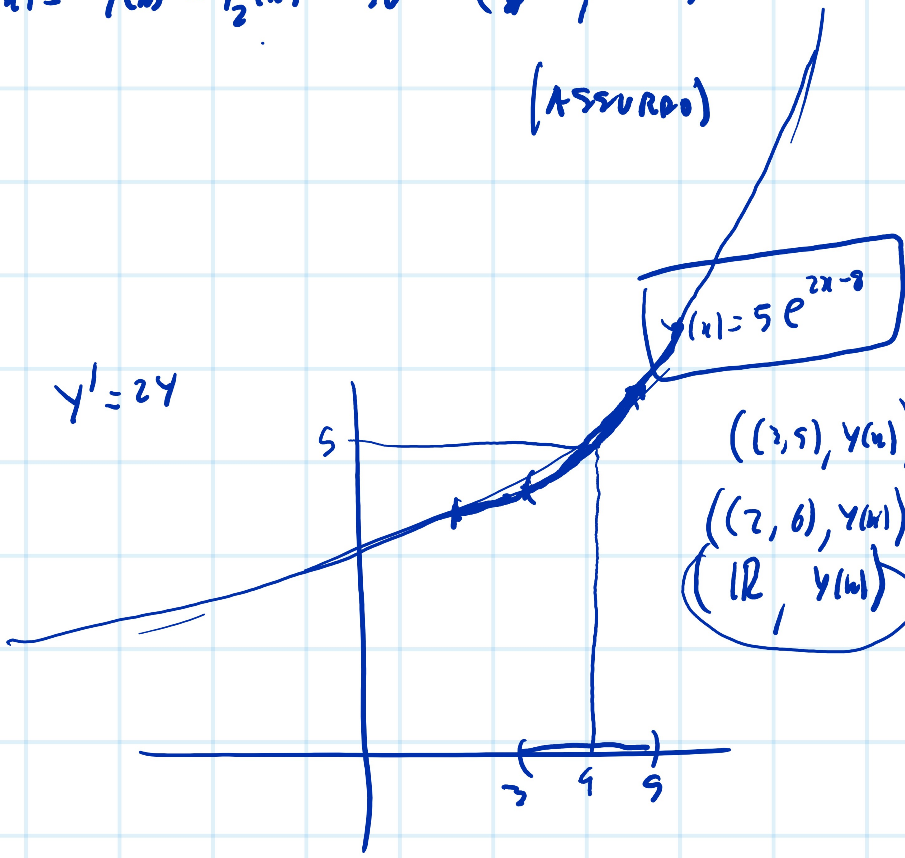
$\exists!\ y(x) \in C^1(I)$ t.c.

È SOL DI $(*)$

$y_1(x) = y(x) = y_2(x)$ su $(\bar{x}-\delta,\ \bar{x}+\delta)$

(ASSURDO)

$y' = 2y$



$((3,5), y(x))$

$((2,6), y(x))$

$(\mathbb{R}, y(x))$

$\boxed{\text{DEF.}}$ DATO $y' = F(x,y)$ (STAND.)

SIANO $\underline{(I_1, y_1(x))}$ (A), $\underline{(I_2, y_2(x))}$ (B) SOL.

DIREMO CHE (A) È PROL. DI (B) SE

$$I_2 \subset I_1 \quad \text{E} \quad y_2(x) = y_1(x) \quad \forall x \in I_2$$

DIREMO INOLTRE CHE (A) È MASSIMALE SE

NON HA ALTRI PROLUNGAMENTI OLTRE LEI STESSA.

$\boxed{\text{TEO.}}$ DATO $(*)\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$ (STAND.)

$\exists$ sempre SOL MASSIMALE.

$\boxed{\text{DIM.}}$

$$\mathcal{F} = \{ (I, y(x)) \mid y(x) \text{ è sol. di } (*) \text{ su } I \}$$

$$I_M = \bigcup_{(I, y(x)) \in \mathcal{F}} I$$

$\forall x \in I_M$ PRENDO $(I_1, y_1(x)) \in \mathcal{F}$ i.e. $x \in I$

$$y_M(x) = \overline{y_1(x)}$$
$$\overline{y_2(x)} \quad (I_2, y_2(x)) \qquad y_2(x) = y_1(x) \text{ su } I_1 \cap I_2 \ni x$$

$$\underline{(I_M, y_M(x))}$$

$$\underline{(I, y(x))} \in \mathcal{J}$$

$$\boxed{y' = \underbrace{f(x)\,\boxed{g(y)}}_{F(x,y)}}$$

Ex 1 : $y' = 2x\,y(y-1)$ $\qquad y(0) = 0, 1, \frac{1}{2}, 2, -1$ $\qquad$ (PAG. 10)

$$\begin{cases} y' = (2x)\cdot\boxed{(y(y-1))} \\ y(0) = \end{cases}$$

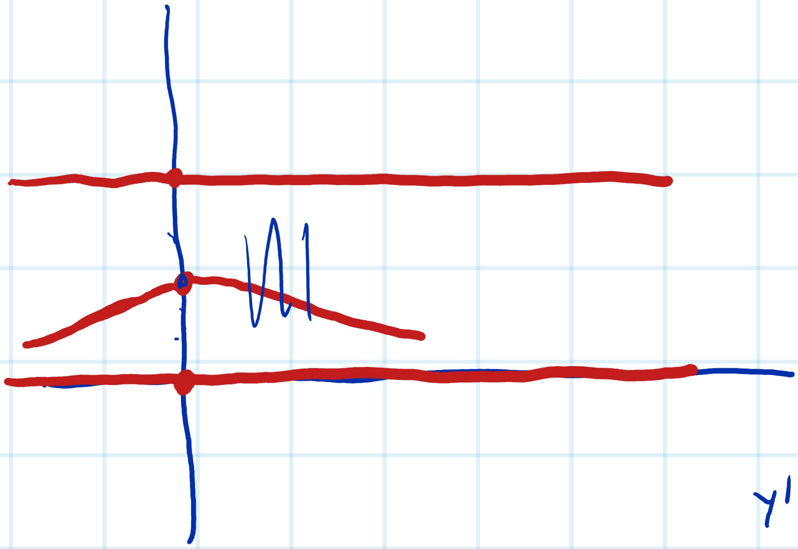
$$\boxed{y(x)\cdot(y(x)-1)}$$

$y=0$

$y=1$

$y(x) \equiv 0$

$y(x) \equiv 1$



NO

$y' = f(x)\cdot g(y)$

$y(x)$ $\qquad \frac{dy}{dx}$

$$y'(x) = 2x\; y(x)\cdot(y(x)-1)$$

$$\boxed{\ln\left(\frac{1}{y(x)}-1\right)} = \int \frac{y'(x)}{y(x)(y(x)-1)}\,dx = \int 2x\,dx$$

$\boxed{y = y(x)} \to \|$ $\qquad = \boxed{x^2 + c}$

$$\int \frac{1}{y(y-1)}\,dy = \int \frac{1}{y-1} - \frac{1}{y}\,dy = \ln(1-y) - \ln y =$$

$$= \ln\left(\frac{1-y}{y}\right) =$$

$$= \ln\left(\frac{1}{y} - 1\right)$$

$$\ln\left(\frac{1}{y(x)} - 1\right) = x^2 + c$$

$$\frac{1}{y(x)} - 1 = e^{x^2+c} \qquad y(x) = \frac{1}{1+e^{x^2}}$$

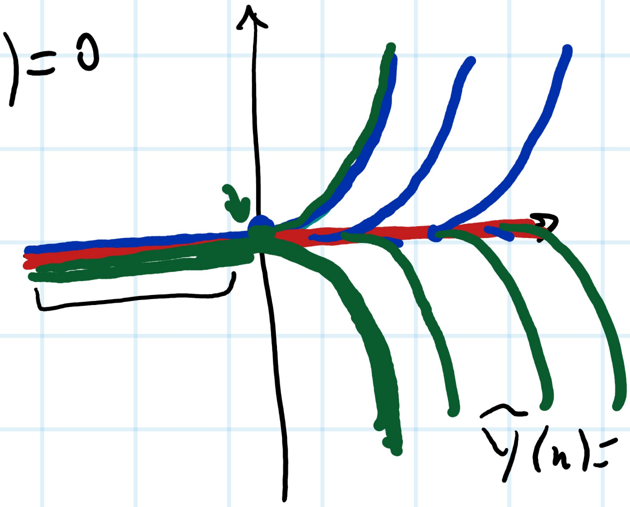
$$\boxed{y(x) = \frac{1}{1+e^{x^2+c}}} \qquad y(0) = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{1+e^{0+c}} \qquad e^c + 1 = 2$$

$$e^c = 1 \qquad c = 0$$

$\boxed{\text{ES. CATTIVO}}$

$$\begin{cases} y' = \sqrt[3]{y} \\ y(0) = 0 \end{cases}$$



$y(x) \equiv 0$

$$\tilde{y}(x) = \begin{cases} 0 & x \le 0 \\ \frac{2\sqrt{2}}{3\sqrt{3}} x\sqrt{x} \end{cases}$$

$y(x) = A x^{\alpha}$

$\alpha - 1 = \frac{\alpha}{3}$

$\frac{2}{3}\alpha = 1 \qquad \alpha = \frac{3}{2}$

$$\hat{y}(x) = \begin{cases} 0 & x \le 0 \\ -\frac{2\sqrt{2}}{3\sqrt{3}} x\sqrt{x} \end{cases}$$

$y(x) = A \cdot x^{\frac{3}{2}}$

$$\boxed{y(x) = \frac{2\sqrt{2}}{3\sqrt{3}} x\sqrt{x}}$$

$\rightarrow y'(x) = \boxed{\frac{3}{2}A}\sqrt{x}$

$\rightarrow \sqrt[3]{y(x)} = \boxed{\sqrt[3]{A}} \cdot \sqrt{x}$

$\frac{3}{2}A = \sqrt[3]{A}$

$A^{\frac{2}{3}} = \frac{2}{3}$

$A = \left(\frac{2}{3}\right)^{\frac{3}{2}} = \boxed{\frac{2\sqrt{2}}{3\sqrt{3}}}$

$$y' = \frac{2\sqrt{2}}{3\sqrt{3}}\left(x^{\frac{3}{2}}\right)' = \frac{\cancel{2}\sqrt{2}}{\cancel{3}\sqrt{3}} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \sqrt{x} = \boxed{\frac{\sqrt{2}}{\sqrt{3}} \cdot \sqrt{x}}$$

$$\sqrt[3]{y(x)} = \sqrt[3]{\frac{2\sqrt{2}}{3\sqrt{3}} x\sqrt{x}} = \boxed{\frac{\sqrt{2}}{\sqrt{3}} \cdot \sqrt{x}}$$

$$\longrightarrow \boxed{\tilde{y}(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{2\sqrt{2}}{3\sqrt{3}} \cdot x\sqrt{x} & x > 0 \end{cases}}$$

$$\tilde{y}'(x) = \frac{\sqrt{2}}{\sqrt{3}} \cdot \sqrt{x} \qquad \boxed{x > 0}$$

$$\tilde{y}'(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{\sqrt{2}}{\sqrt{3}} \sqrt{x} & x > 0 \end{cases}$$

$$\Rightarrow y' = \sqrt[3]{y}$$

$y' = f(y)$ E.Q. AUTONOME

$\boxed{y(x)}$ È SOL. $\Longrightarrow \forall \tau \in \mathbb{R}$ $\underbrace{\boxed{y(x-\tau)}}_{z(x)}$ È SOL.

$$\boxed{z'(x)} = \left(y(x-\tau)\right)' = y'(x-\tau) \cdot 1 = f\left(y(x-\tau)\right) = f$$
$$= \boxed{f(z(x))}$$

$\boxed{\text{T. Prolung. fuori dai Compatti}}$ $F : \Omega \to \mathbb{R}$

DATO $(*)$ $\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$ (IP. STANDARD)

SIA $K \subset \Omega$ COMPATTO E SIA

$((a,b), y(x))$ SOL DI $(*)$ T.C.

$\forall x \in (a,b) \quad (x, y(x)) \in K$.

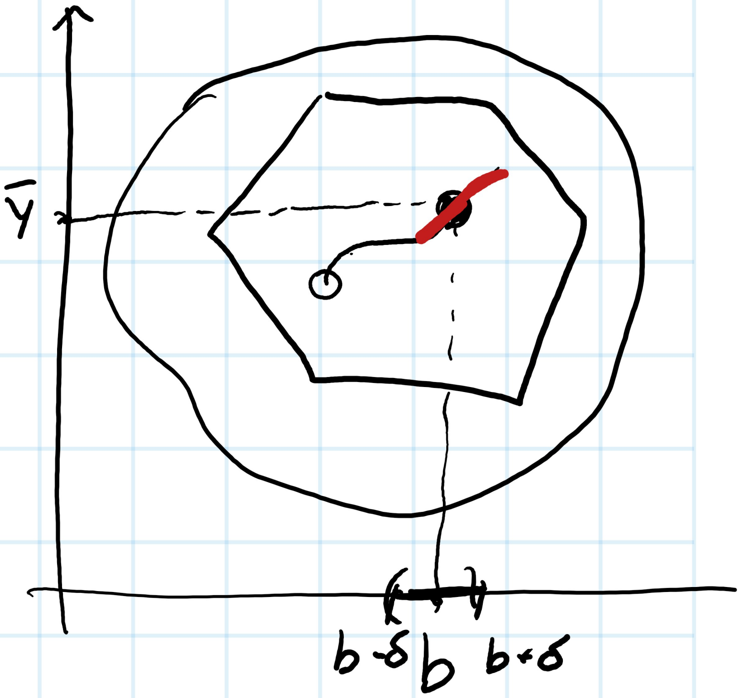
ALLORA $\exists\, \delta > 0$ e $\tilde{y}(x) \in C^1((a-\delta, b+\delta))$

t.c. $((a-\delta, b+\delta), \tilde{y}(x))$ È PROLUNGAMENTO

DI $((a,b), y(x))$.

$\boxed{\text{DIM.}}$

$$M = \max \left\{ |F(x,y)| \;\middle|\; (x,y) \in K \right\}$$



$$y'(x) = F(x, y(x))$$

$$\forall x \in (a,b) \quad |y'(x)| \leq M$$

$$\bar{y} = \lim_{x \to b^-} y(x) \in \mathbb{R}$$

$$\begin{cases} y' = F(x,y) \\ y(b) = \bar{y} \end{cases}$$

$$\exists\, \delta > 0 \quad \exists!\, \boxed{y_1(x)} \in C_1((b-\delta, b+\delta))$$

$$\boxed{(a, b+\delta)} \qquad \boxed{\tilde{y}(x)} = \begin{cases} y(x) & x \in (a,b) \\ y_1(x) & x \in [b, b+\delta) \end{cases}$$

$$\lim_{x\to b^-} \tilde{y}(x) = \lim_{x\to b^-} y(x) = \bar{y} = \underline{y_1(b)} = \underline{\lim_{x\to b^+} y_1(x)}$$

$$\lim_{x\to b^-} \tilde{y}'(x) = \lim_{x\to b^-} \underline{y'(x)} = \lim_{x\to b^-} F(x, y(x)) = \boxed{F(b, \bar{y})}$$

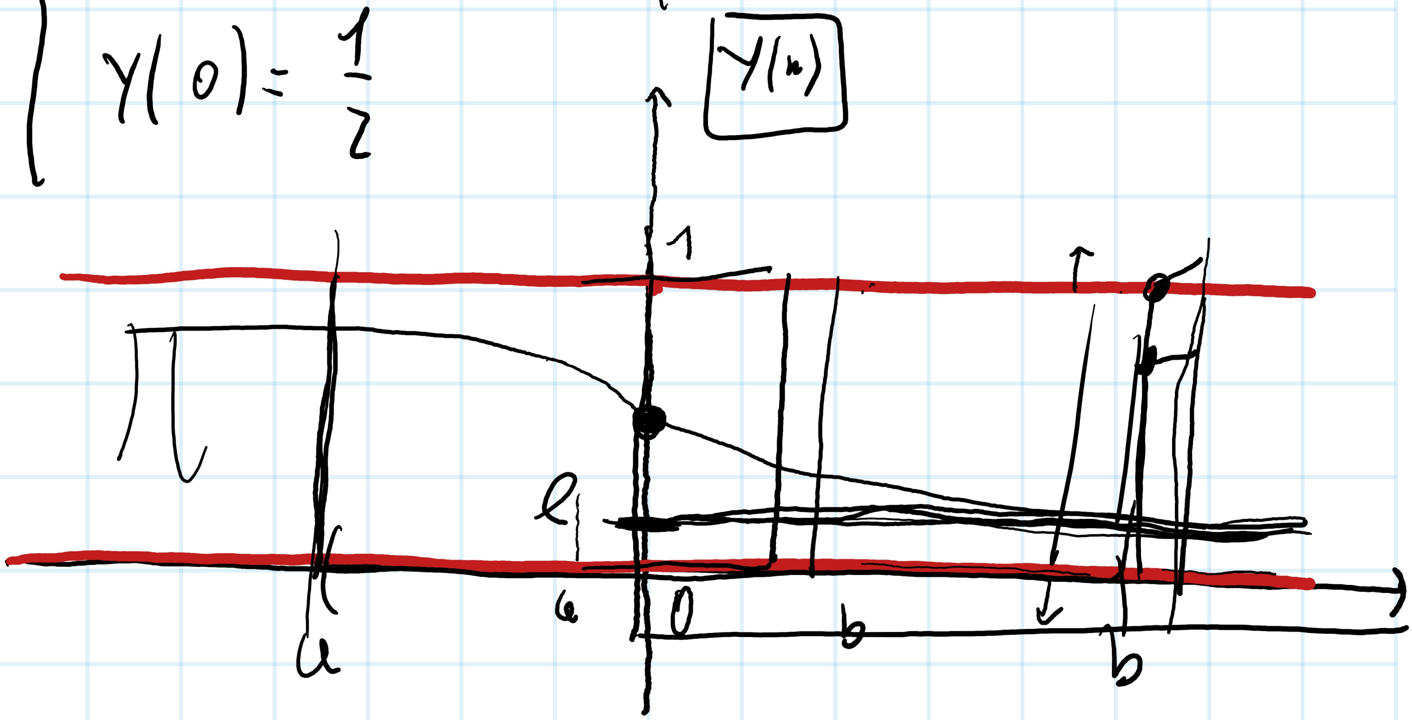
$$\lim_{x\to b^+} \tilde{y}'(x) = \lim_{x\to b^+} \underline{y_1'(x)} = \lim_{x\to b^+} F(x, y_1(x)) = \boxed{F(b, \bar{y})}$$

$$\lim_{x\to b^+} \frac{\tilde{y}(x) - \tilde{y}(b)}{x-b} = \lim_{x\to b^+} \frac{y_1(x) - y_1(b)}{x-b} = \lim_{x\to b^+} y_1'(\xi_x) =$$

$$b < \xi_x < x$$

$$= y_1'(b) = F(b, y_1(b)) = \boxed{F(b, \bar{y})}$$

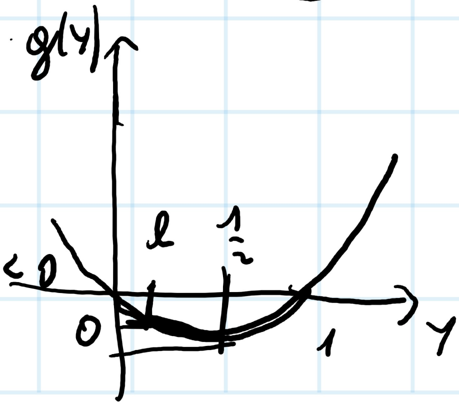
$$\begin{cases} y' = \arctan\big(y(y-1)\big) \\ y(0) = \dfrac{1}{2} \end{cases}$$



$$K = \boxed{[a,b]\times[0,1]}$$

$$\lim_{x\to+\infty} y(x) = l > 0 \qquad \forall x\in[0,+\infty) \quad l < y(x) \le \frac{1}{2}$$

$$\arctan\big(\overbrace{y(y-1)}^{g(y)}\big)$$



$$\arctan\big(y(y-1)\big) < \arctan\big(l(l-1)\big) < 0$$

$$\boxed{\forall x \geq 0}$$

$$\underbrace{\arctan\big(y(x)(y(x)-1)\big)} \leq \underbrace{\arctan\big(l(l-1)\big)}_{\boxed{C}} \boxed{<0}$$

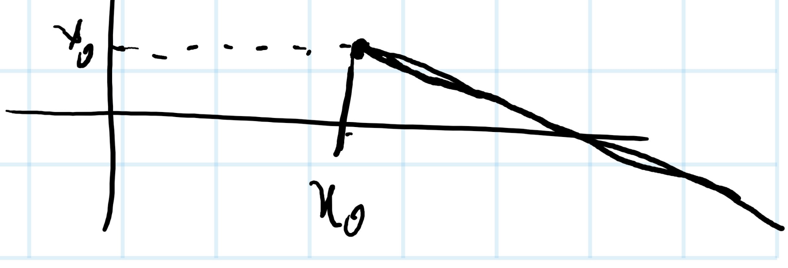
$$\lim_{x \to +\infty} y(x) = 0 \ ?$$

$$\lim_{x \to +\infty} y(x) = \boxed{l} \begin{array}{l} < \frac{1}{2} \\ \geq 0 \end{array}$$

$$0 < l < \frac{1}{2}$$

$$\lim_{x \to +\infty} y'(x) = \lim_{x \to +\infty} \arctan\big(y(x)(y(x)-1)\big) = \boxed{\arctan\big(l(l-1)\big)} < 0$$

$\exists \lambda < 0$ t.q. [illegible] x $y'(x) < \lambda$



$\lambda < 0$

$$y = \left(y_0 + \lambda(x - x_0)\right)$$

$$D(x) = \boxed{y(x)} - \boxed{\left(y_0 + \lambda(x - x_0)\right)} \qquad D(x_0) = 0$$

$$D'(x) = y'(x) - \lambda < 0$$

$$D(x) \leq 0 \qquad x > x_0$$

$$\underbrace{y(x)}_{\downarrow \; -\infty} < \underbrace{y_0 + \lambda(x + x_0)}_{\downarrow \; -\infty} \qquad x > x_0$$

$$y(x) = \underbrace{\arctan\left(y(x)\left(y(x) - 1\right)\right)}$$

$$\boxed{g(x) = \boxed{\frac{1}{2}}} \qquad g'(x) = 0$$

$$g'(x) > \underbrace{\arctan\left(g(x)\left(g(x) - 1\right)\right)}_{\arctan\left(-\frac{1}{4}\right)}$$

$y(x) = $

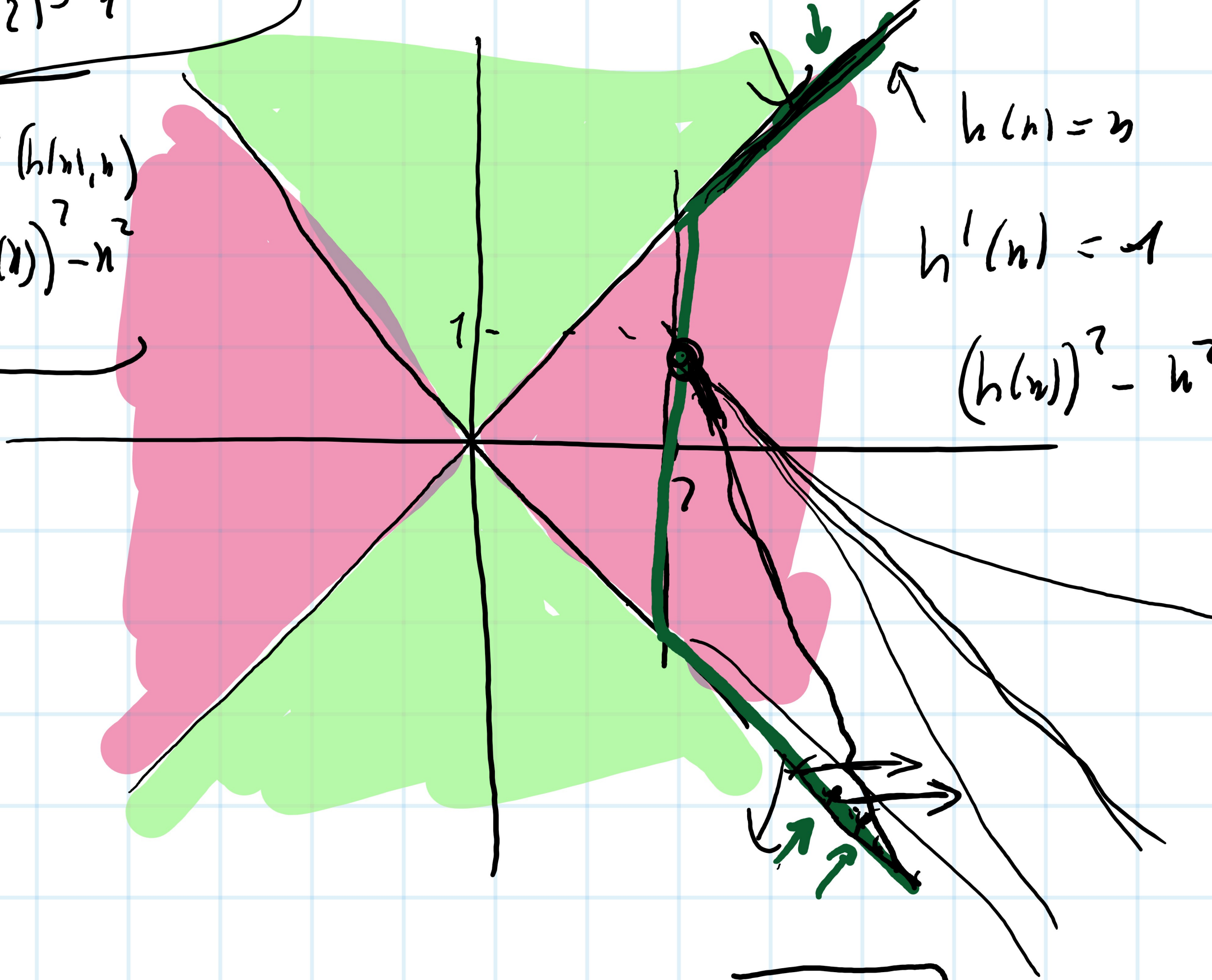
$$\begin{cases} y' = \boxed{y^2 - x^2} \\ y(2) = 1 \end{cases}$$

$$y'(x) = (y(x))^2 - x^2$$

$$1^2 - 2^2 = -2$$

$F(h(x), x)$

$$h'(x) > (h(x))^2 - x^2$$



$h(x) = x$

$h'(x) = 1$

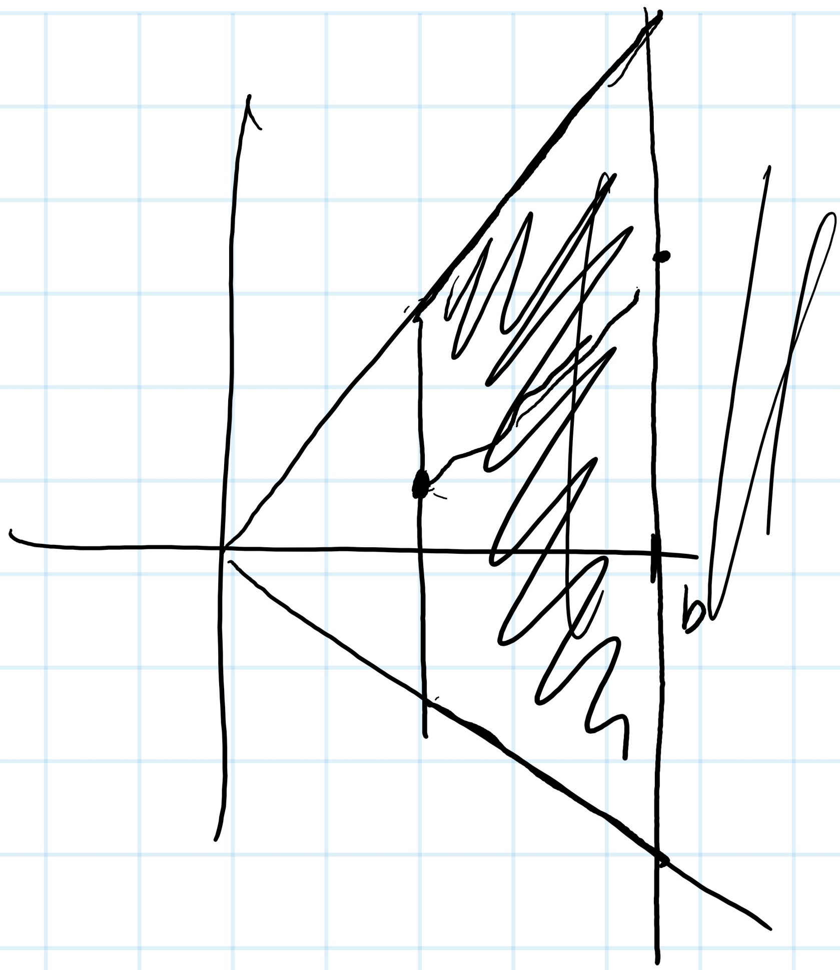
$(h(x))^2 - x^2 = 0$

$$\boxed{y^2 - x^2} = 0$$

$$g(x) = -x$$

$$\boxed{g'(x) = -1}$$

$$\underbrace{\boxed{g'(x)}}_{-1} < \overbrace{\underbrace{(g(x))^2 - x^2}_{0}}^{F(g(x),x)}$$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