A.M. 2 - LEZ. EQD. 03 - STUDIO QUALITATIVO - (26/05/2023)

.... DA LEZ. SCORSA

0) UTILIZZO "PULITO" DI PROLUNGABILITÀ FUORI DAI COMPATTI IN $y' = \arctan(y(y-1))$ $y(0) = \frac{1}{2}$ + DIMO PULITA CHE $y(x) \to 0$ PER $x \to +\infty$.

1) SOPRASOLUZIONE E SOTTOSOLUZIONE
   a) DEFINIZIONE (PAG. 65)
   b) TEOREMI (PAG 66-67)

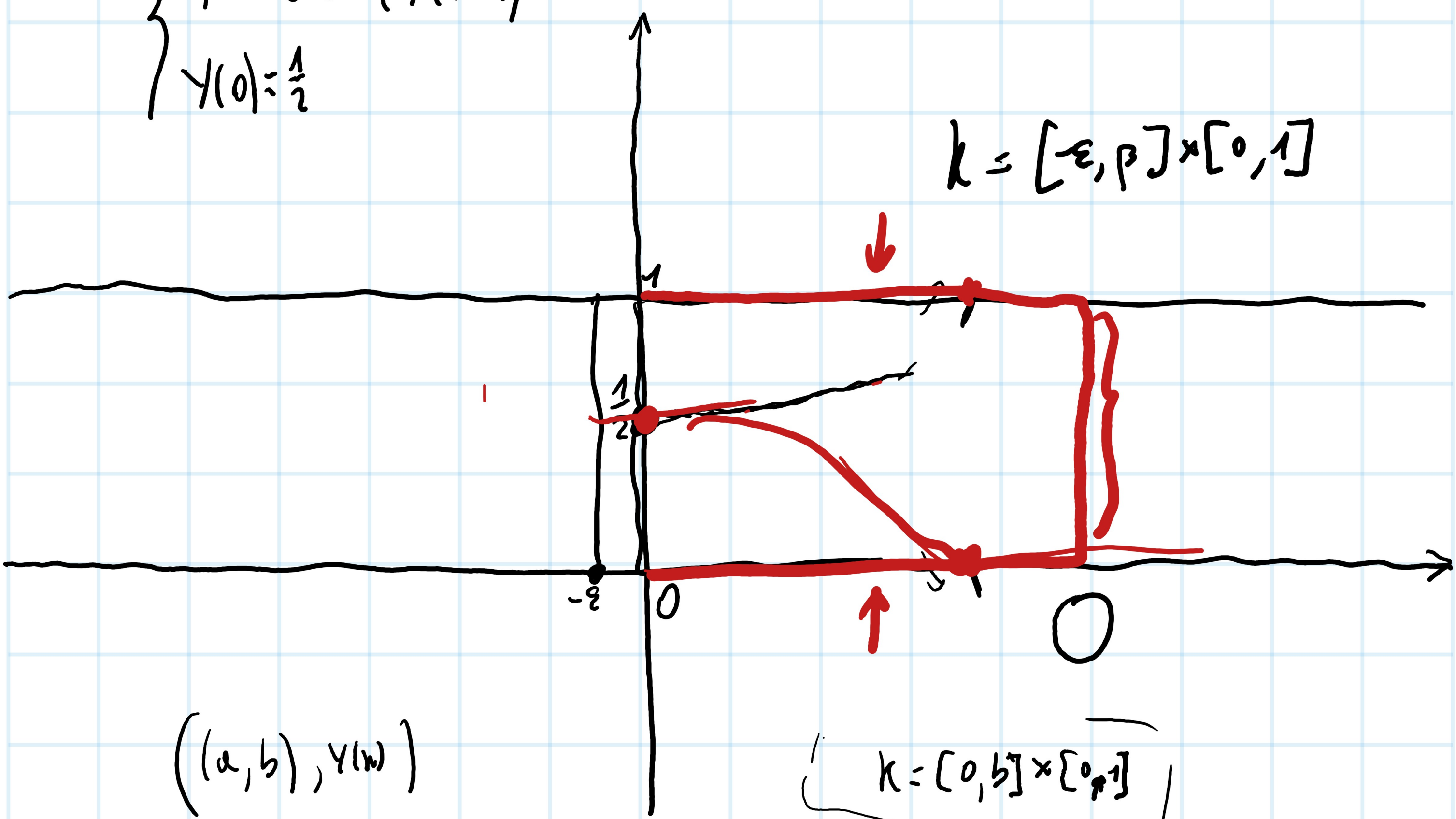
2) TEO. CONFRONTO (PAG. 67-68)

3) STUDIO QUALITATIVO DI $\begin{cases} y' = y^2 - x^2 \\ y(0) = \alpha \end{cases}$ (PAG 68-71)

4) VERSIONE DEBOLE DI 1b (NUOVO) NON FATTA

$$\begin{cases} y' = \arctan(y(y-1)) \\ y(0) = \frac{1}{2} \end{cases}$$

$k = [-\varepsilon, \beta] \times [0,1]$



$k = [0,b] \times [0,1]$

$((a,b), y(x))$

$a = -\infty$

$b = +\infty$

$((0,b), y(x))$

$((0,b+\delta), y(x))$

$((a, b+\delta), y(x))$

$\boxed{OSS}$ DATO $\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$ (P. STANDARD)

$K \subset \Omega$ COMPATTO $(x_0, y_0) \in \mathring{K}$

ALLORA SE $((a,b), y(x))$ È SOL MASSIMALE.

ALLORA $\exists^{\text{no}}$ $\bar{x}_1 \in (a, x_0)$ $\bar{x}_2 \in (x_0, b)$ t.c.

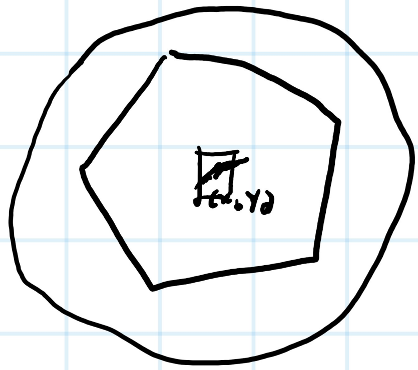
T.C. $\underbrace{(\bar{x}_1, y(\bar{x}_1)) \in \partial k}$.

DIM

$\rightarrow A^+ = \{ x \in (x_0, b) \mid \underline{(x, y(x))} \in \overset{\circ}{k} \}$

$\sup A^+ = \bar{x}_1$

I° $\boxed{\bar{x}_1 = b}$ $\bar{y} = \lim_{x \to \bar{x}_1^-} y(x)$ $\nexists$



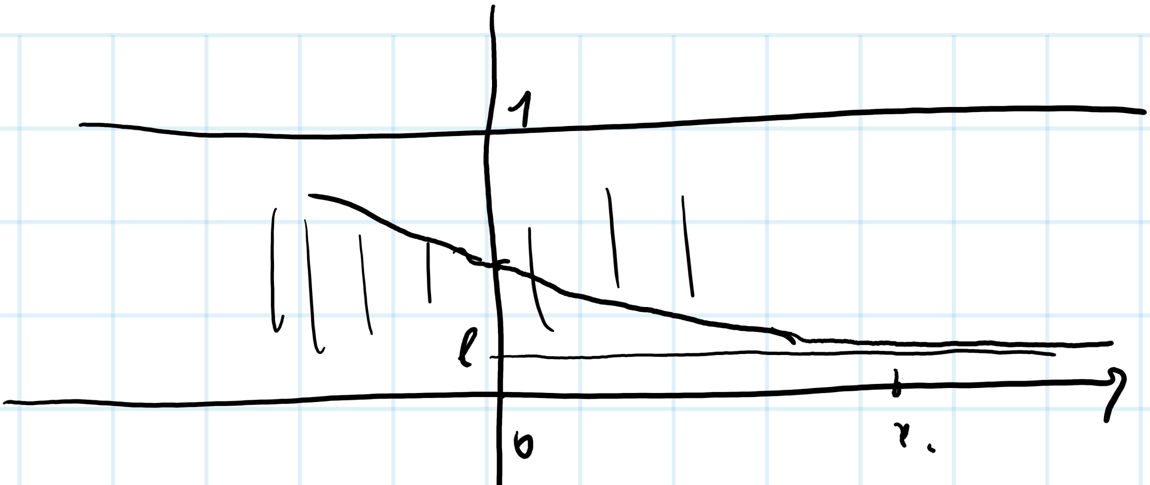
$(\bar{x}_1, \bar{y})$

II° $\bar{x}_1 < b$

$\{ x \in (x_0, b) \mid (x, y(x)) \notin \overset{\circ}{k} \}$

$\forall \varepsilon > 0 \ \exists x \in (\bar{x}_1, \bar{x}_1 + \varepsilon)$ t.c. $\underline{(x, y(x)) \notin \overset{\circ}{k}}$

$\underline{(\bar{x}_1, y(\bar{x}_1))} \in \partial k$



$$\underbrace{\lim_{x\to+\infty} y(x) = \ell \in \left[0, \tfrac{1}{2}\right)}$$

$\boxed{\text{P.A. } \ell > 0}$ $\boxed{\ell \in \left(0, \tfrac{1}{2}\right)}$

$$\lim_{x\to+\infty} y'(x) = \lim_{x\to+\infty} \arctan\Big(\underbrace{y(x)}_{\downarrow\ \ell}\big(\underbrace{y(x)-1}_{\downarrow\ \ell-1}\big)\Big) = \underbrace{\arctan\big(\ell(\ell-1)\big)} = c < 0$$

$$y'(x) = \underbrace{\arctan\big(y(x)(y(x)-1)\big)}$$

$$\underbrace{\exists\, x_0 > 0 \text{ t.c. } x > x_0 \quad y'(x) \le \frac{c}{2} < 0} \Rightarrow y(x) \to -\infty$$

$$\forall x > x_0 \quad \boxed{y(x)} = \boxed{y(x) - y(x_0) +} \, y(x_0) = y(x_0) + \int_{x_0}^{x} y'(t)\,dt \le$$

$$\le y(x_0) + \int_{x_0}^{x} \frac{c}{2}\,dt = \boxed{y(x_0) + \frac{c}{2}(x - x_0)} \to -\infty$$

$(*)$

$\boxed{\text{DEF.}}$ DATO $\quad y' = F(x,y)$ $\quad \boxed{\text{IP. STANDARD}}$

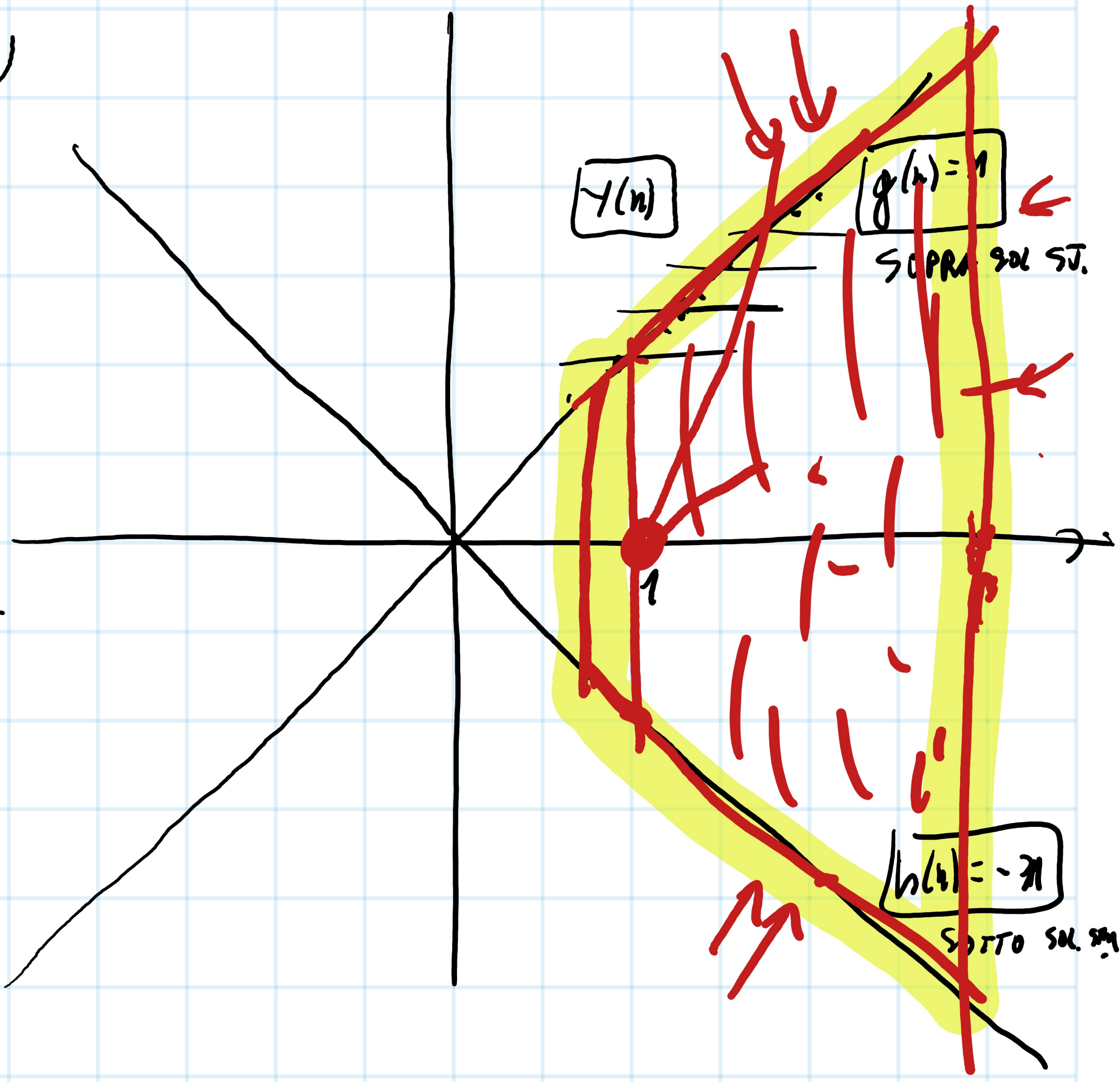
$F : \Omega \to \mathbb{R}$

DATA $\left((a,b),\ g(x)\right)$ DIREMO CHE È SOPRA-SOL. (STRETTA) DI $(*)$
(SOTTO-SOL) (STR)

SE $\forall x \in (a,b)$ $\quad (x, g(x)) \in \Omega$ E

$$g'(x) \geq F(x, g(x)) \quad (>)$$

$$\begin{cases} y' = y^2 - x^2 \\ y(1) = 0 \end{cases}$$



$g'(x) = 1$

$F(g(x), x) =$

$= (g(x))^2 - x^2 =$

$= x^2 - x^2 =$

$= 0$

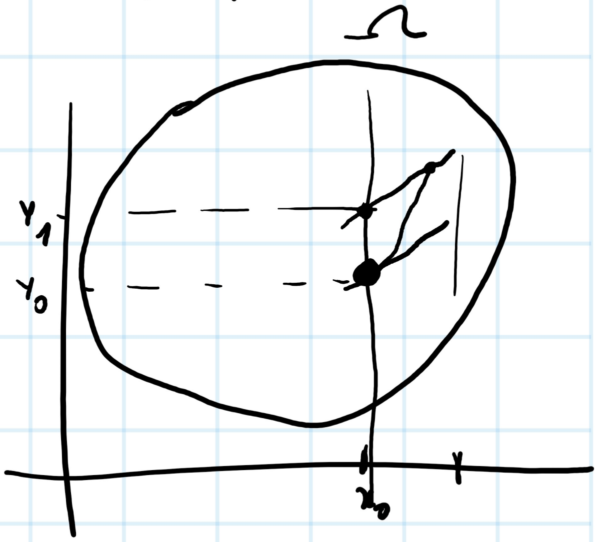
$\boxed{\text{T. SOPRA-SOL.}}$

$F: \Omega \to \mathbb{R}$

DATA $y' = F(x,y)$ (CON IP. STANDARD)

SIANO $(x_0, y_0) \in \Omega$ con $y_0 < y_1$

$(x_0, y_1) \in \Omega$



SIA $((a,b), y(x))$ SOL. DI

$$\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

E SIA $((a,b), g(x))$

SOPRA-SOL DI $\boxed{y' = F(x,y)}$ t.c. $g(x_0) = y_1$
(STRETTA)

ALLORA

$\forall x \in [x_0, b) \quad y(x) < g(x)$

$\boxed{\text{DIM}}$

P.A. $\underline{A^+ = \{ x > x_0 \mid y(x) = g(x) \}}$

$\underline{\inf} A^+ = \bar{x} \in \mathbb{R} \qquad \bar{x} \geq x_0$

$D(x) = y(x) - g(x)$ VALE 0 IN $\bar{x}$ PERCHÈ SE P.A.

FOSSE $D(\bar{x}) > 0 \Rightarrow D(x) > 0$ IN $(\bar{x}-\delta, \bar{x}+\delta) \Rightarrow$

IN $[\bar{x}, \bar{x}+\delta)$ NON CI SONO PUNTI

DI $A^+ \Rightarrow \bar{x}$ NON È INF.

$\boxed{\text{ASSU.}}$

QUINDI $y(\bar{x}) = g(\bar{x})$

E $y(x) < g(x)$ SE $x \in [x_0, \bar{x})$

ALLORA $\forall x \in [x_0, \bar{x})$ SI HA

$$\frac{y(x) - y(\bar{x})}{x - \bar{x}} \geq \frac{g(x) - g(\bar{x})}{x - \bar{x}}$$

$y = F(y, x)$

$y(x) < g(x)$

$g'(\bar{x}) > F(g(\bar{x}), \bar{x})$

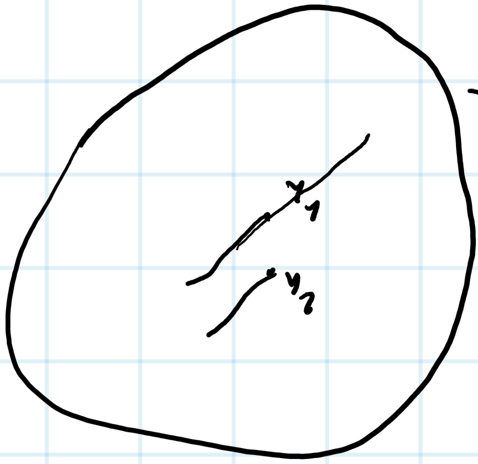
$$\boxed{\frac{y(x) - y(\bar{x})}{x - \bar{x}}} > \boxed{\frac{g(x) - g(\bar{x})}{x - \bar{x}}}$$

$$F(y(\bar{x}), \bar{x}) = \boxed{y'(\bar{x})} = \lim_{x \to \bar{x}} \blacksquare \geq \lim_{x \to \bar{x}} \blacksquare = \boxed{g'(\bar{x})} > F(g(\bar{x}), \bar{x})$$

$\boxed{\text{T. CONFRONTO}}$ $\qquad \forall (x,y) \in \Omega \qquad F(x,y) < G(x,y)$

$$\begin{cases} y' = \boxed{F(x,y)} \\ y(x_0) = y_1 \end{cases} \qquad \begin{cases} y' = \boxed{G(x,y)} \\ y(x_0) = y_2 \end{cases} \qquad (\text{IP. ST})$$

$$\Big((a,b)\,,\, y_1(x) \Big) \qquad\qquad \boxed{\Big((a,b)\,,\, y_2(x) \Big)}$$



Ω

ALLORA

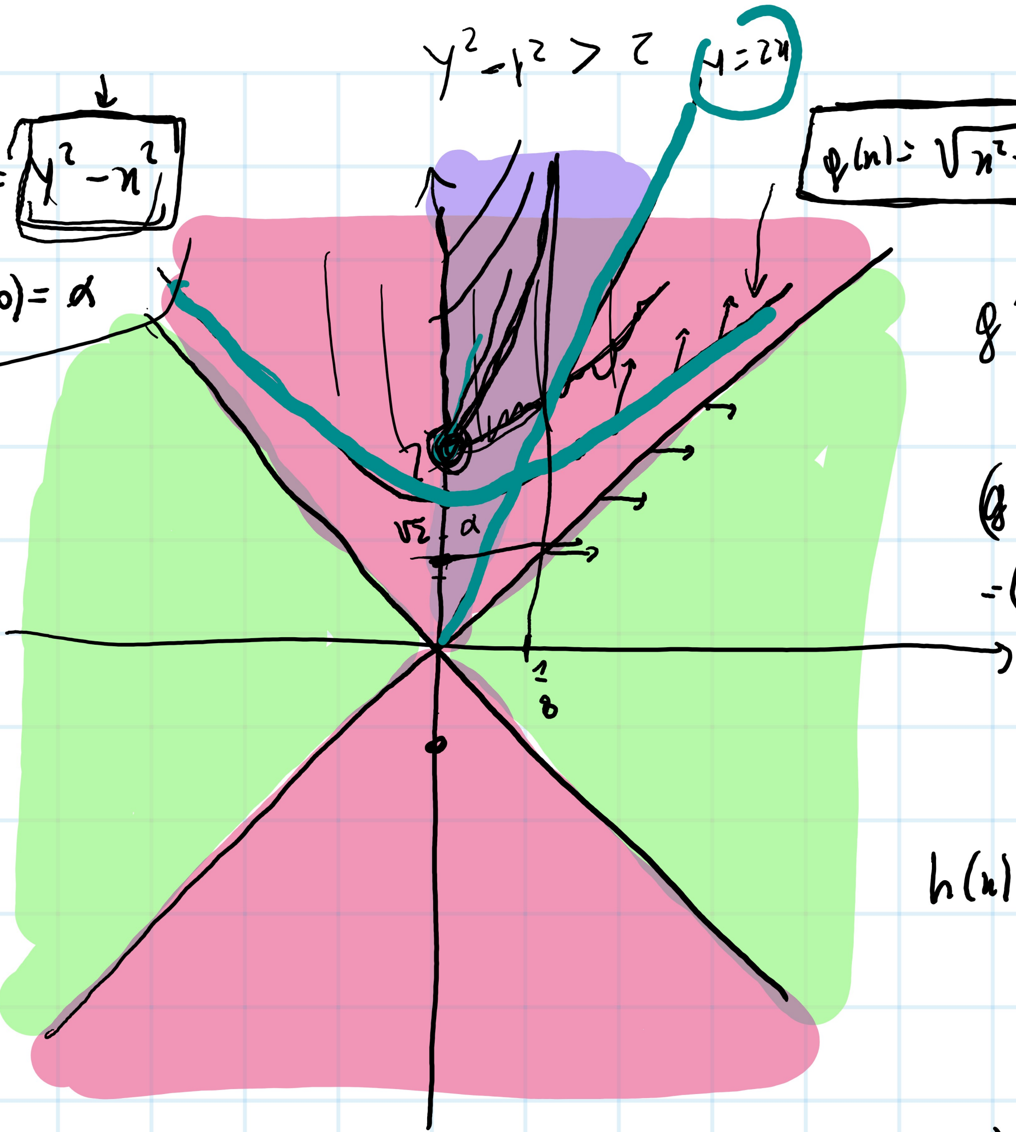
$$y_1 < y_2 \implies y_1(x) < \boxed{y_2(x)} \qquad \forall x > x_0$$

$$y_2 < y_1 \implies y_2(x) < y_1(x) \qquad \forall x < x_0$$

DIM

$$y_2'(x) = G(y_2(x), x) > F(y_2(x), x)$$

$$\begin{cases} y' = \boxed{y^2 - x^2} \\ y(0) = \alpha \end{cases}$$



$$\boxed{\varphi(x) = \sqrt{x^2+2}}$$

$$g' = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+2}}$$

$$(g(x))^2 - x^2 =$$
$$= (\sqrt{x^2+2})^2 - x^2 =$$
$$= 2$$

$$h(x) = x$$

$$g'(x) = (\sqrt{x^2+2})' =$$
$$= \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+2}} =$$
$$\boxed{\frac{x}{\sqrt{x^2+2}}}$$

$$y^2 - x^2 = 2$$
$$y^2 = x^2 + 2$$
$$y = \pm\sqrt{x^2+2}$$

$$\boxed{y > 2x}$$

$y(x)$

$$y^2 - x^2 = \left(\frac{y}{2} + \frac{y}{2}\right)^2 - x^2 >$$

$$> \frac{y^2}{4} + \boxed{\frac{y^2}{4}} - x^2 > \frac{y^2}{4} + x^2 - x^2 = \frac{y^2}{4}$$

$$\left(\frac{y}{2}\right)^2 > x^2$$

$G(x,y) = y^2 - x^2$

$F(x,y) = \dfrac{y^2}{4}$

$G(x,y) > F(x,y)$

$$\Omega = \{(x,y) \mid x > 0,\ y > 2x\}$$

$$\underset{\frac{y^2}{4}}{F(x,y)} < \underset{y^2 - x^2}{G(x,y)}$$

$$\begin{cases} y' = y^2 - x^2 \\ y(0) = 2,1 \end{cases}$$

$$\boxed{\begin{cases} y' = \dfrac{y^2}{4} \\ y(0) = 2 \end{cases}}$$

$$4\left(-\frac{1}{y}\right) = x + c$$

$$y' = \frac{y^2}{4} \qquad \int \frac{4y'(x)}{y^2(x)} = \int 1.$$

$$y(x) = -\frac{\frac{1}{4}}{x + c}$$

$$y = -\frac{\frac{1}{4}}{x - \frac{1}{8}}$$

$$y(0) = 2$$

$$2 = \frac{-\frac{1}{4}}{c} \qquad c = -\frac{1}{8}$$

$$\boxed{y(x) = -\frac{2}{8x-1}}$$