Lezione 1: Insiemi e loro operazioni

INDICE

-1 **PRESENTAZIONE CORSO** (DOCENTI - CALENDARIO - MAT. DIDATTICO - ESAMI)

0 **AVVERTIMENTO!** (ELEMENTARE V.S. FACILE)

1 INSIEMI SOLO COME LINGUAGGIO

1.1 DEFINIZ: $\emptyset$, $\subset$, $=$, DISGIUNTI

1.2 NOTAZIONI: $\{ \cdots$ ELENCO $\cdots \}$, $\{x \in A \mid \cdots$ CONDIZIONE $\}$, $\{f(n) \mid n \in A\}$

1.3 $\mathcal{P}(\ldots)$, $\cup$, $\cap$, $-$, $\times$, COMPLEMENTARE

1.4 DISTRIBUTIVITÀ

1.5 DE MORGAN

1.6 $\cup$ E $\cap$ INFINITE

1.7 ESEMPI DI INSIEMI NUMERICI (SOLO DEF. INFORMALE, DOMANI QUELLA FORMALE)

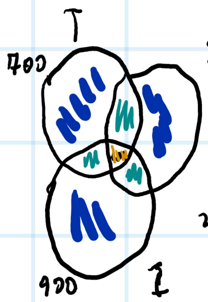
1.8 ESERCIZI

ES.1 IN UNA SCUOLA OGNI STUDENTE PARLA ALMENO UNA DELLE 3 LINGUE INGLESE, FRANCESE, TEDESCO. INOLTRE SAPPIAMO CHE:

1) IL TEDESCO È PARLATO DA 700 STUDENTI
2) IL FRANCESE .... 800 ......
3) L'INGLESE .... 900 ......
4) SOLO 100 STUDENTI PARLANO 3 LINGUE
5) 1200 STUDENTI PARLANO UNA SOLA DELLE 3 LINGUE

QUANTI SONO IN TUTTO GLI STUDENTI?

PER CASA
ESERCIZI
19 – 30
PAG. 10
ESERCIZIARIO



1700 + 490 1100

$\boxed{1750}$

800 + 700 + 900

$2400 = \overbrace{\text{BLU}}^{1200} + \overbrace{2 \text{ verdi}}^{x} + \overbrace{3 \text{ gialli}}^{100}$

$\boxed{490}$

ES.2 SIA $\mathcal{J} = \{ I = [a,b] \mid a,b \in \{0,1,\ldots,5\}, b-a=3 \}$ TROVARE:

$$\bigcup_{(I,J) \in \mathcal{J} \times \mathcal{J}} (I \cap J) \quad \text{E} \quad \bigcap_{(I,J) \in \mathcal{J} \times \mathcal{J}} (I \cup J)$$

ES.3 COME [ES.2] MA CON $\mathcal{J} = \{ I \in [a,b] \mid a,b \in [0,5], b-a=3 \}$

ES.4 DIRE SE SONO UGUALI I 2 INSIEMI:

$$\bigcup_{n \in \mathbb{N}} \left(\bigcap_{k \in \mathbb{N}} \left(\frac{1}{n+1} - \frac{1}{k+1}, \frac{1}{n+1} + \frac{1}{k+1} \right) \right) \quad \text{E} \quad \bigcap_{k \in \mathbb{N}} \left(\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1} - \frac{1}{k+1}, \frac{1}{n+1} + \frac{1}{k+1} \right) \right)$$

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STUDIO 1009

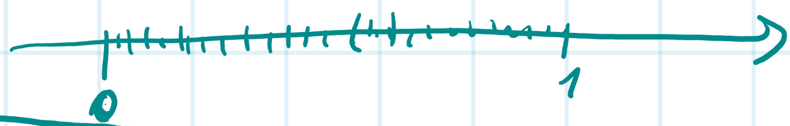
ESEMPIO DI PROBLEMA ELEMENTARE MA NON SEMPLICE

$A = \{ \sqrt{n} - \lfloor \sqrt{n} \rfloor \mid n \in \mathbb{N} \}$ È DENSO IN $[0,1]$

$\forall (a,b) \subset [0,1] \ \exists x \in A$ t.c. $x \in (a,b)$

$\sqrt{x} - \lfloor \sqrt{x} \rfloor = \{\sqrt{x}\}$

PARTE DECIMALE DI $\sqrt{x}$



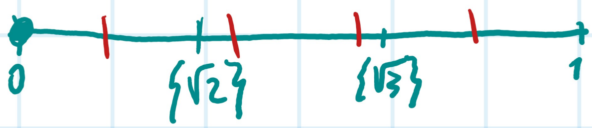
$\{\sqrt{0}\}$

$\{\sqrt{1}\}$

$\{\sqrt{2}\}$

$\{\sqrt{3}\}$

$\{\sqrt{4}\}$



$\overbrace{\{\sqrt{5}\} \quad \{\sqrt{6}\} \quad \{\sqrt{7}\} \quad \{\sqrt{8}\}}$ $\{\sqrt{9}\}$

$n = k^2$

k^2+1

k^2+2

$\vdots$

$n = k^2 + \underbrace{2k} = k^2+2k+1-1 = (k+1)^2 - 1$

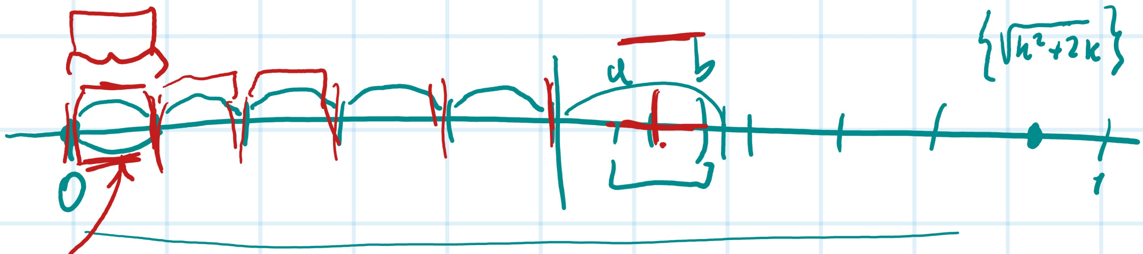
$k^2 + p$

$\underbrace{0 \le p \le 2k}$

$\lfloor \sqrt{k^2+p} \rfloor = k$

$\{\sqrt{k^2+p}\} = \sqrt{k^2+p} - \overbrace{\lfloor\sqrt{k^2+p}\rfloor}^{k} =$

$= \sqrt{k^2+p} - k$



$\{\sqrt{k^2}\}, \{\sqrt{k^2+1}\}, \boxed{\dots}, \boxed{\{\sqrt{k^2+p}\}}, \dots, \{\sqrt{k^2+2k}\}$

$\frac{1}{\sqrt{k^2+1}+\sqrt{k^2}} \le \boxed{\frac{1}{k}} \leftarrow b-a$

$\boxed{\{\sqrt{k^2+p}\} - \{\sqrt{k^2+p-1}\}} = \boxed{\frac{1}{\sqrt{k^2+p}+\sqrt{k^2+p-1}}}$

$= (\sqrt{k^2+p} - \cancel{k}) - (\sqrt{k^2+p-1} - \cancel{k}) =$

$= \frac{\sqrt{k^2+p}-\sqrt{k^2+p-1}}{\sqrt{k^2+p}+\sqrt{k^2+p-1}} \cdot (\sqrt{k^2+p}+\sqrt{k^2+p-1}) =$

$$= \frac{\cancel{k^2}+\cancel{p}-(\cancel{k^2}+\cancel{p}-1)}{\sqrt{k^2+p}+\sqrt{k^2+p-1}} = \frac{1}{\sqrt{k^2+p}+\sqrt{k^2+p-1}}$$

RIASSUMENDO:

I CALCOLI SOPRA MOSTRANO CHE, PER OGNI INTERO POSITIVO k, SI HA

(*) $$0=\{\sqrt{k^2}\}<\{\sqrt{k^2+1}\}<\dots<\{\sqrt{k^2+p}\}<\dots<\{\sqrt{k^2+2k}\}<1$$

E CHE GLI INTERVALLI IN CUI ESSI SUDDIVIDONO $[0,1]$ SONO DI LUNGHEZZA DECRESCENTE, OVVERO:

$$\{\sqrt{k^2+1}\}-0>\{\sqrt{k^2+2}\}-\{\sqrt{k^2+1}\}>\dots\dots>\{\sqrt{k^2+p}\}-\{\sqrt{k^2+p-1}\}>\dots\dots>1-\{\sqrt{k^2+2k}\}$$

DI CONSEGUENZA, COMUNQUE SIA PRESO (a,b) TALE CHE $0\leq a<b\leq 1$, BASTA PRENDERE k IN MODO CHE

$\{\sqrt{k^2+1}\}-0<(b-a)$ PER ESSERE SICURI CHE ALMENO UNO DEI PUNTI (*) STA DENTRO (a,b)

INSIEMI

DEF. Un insieme A si dice vuoto se nessun $x \in A$.

DEF. 2 Dati A, B insiemi $A \subset B$ significa che $x \in A \Rightarrow x \in B$

$A = B$ significa $A \subset B$ e $B \subset A$

DEF. 3 Dati A, B insiemi definiamo

$A \cup B =$ l'insieme di tutti gli elementi che app. ad A o a B

$A \cap B =$ . . . . . . A e B

$A \setminus B =$ ad A ma non a B

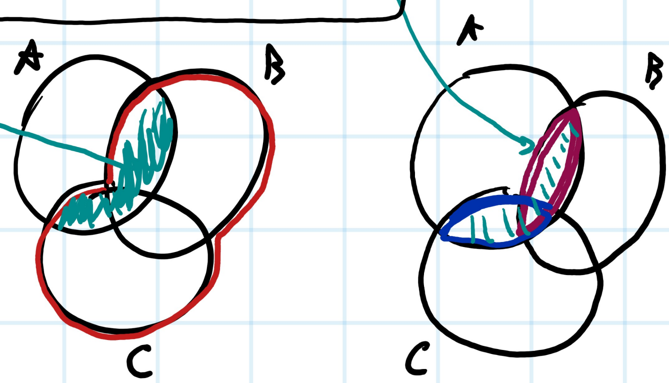
$\mathcal{P}(A) = \{ I \mid I \subset A \}$

$A \times B = \{ (a,b) \mid a \in A, b \in B \}$

P. 1 DISTRIBUTIVITÀ

$$\rightarrow \underbrace{A \cap (B \cup C)}_{} \overset{\downarrow}{=} \underbrace{\boxed{(A \cap B) \cup (A \cap C)}}_{}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$



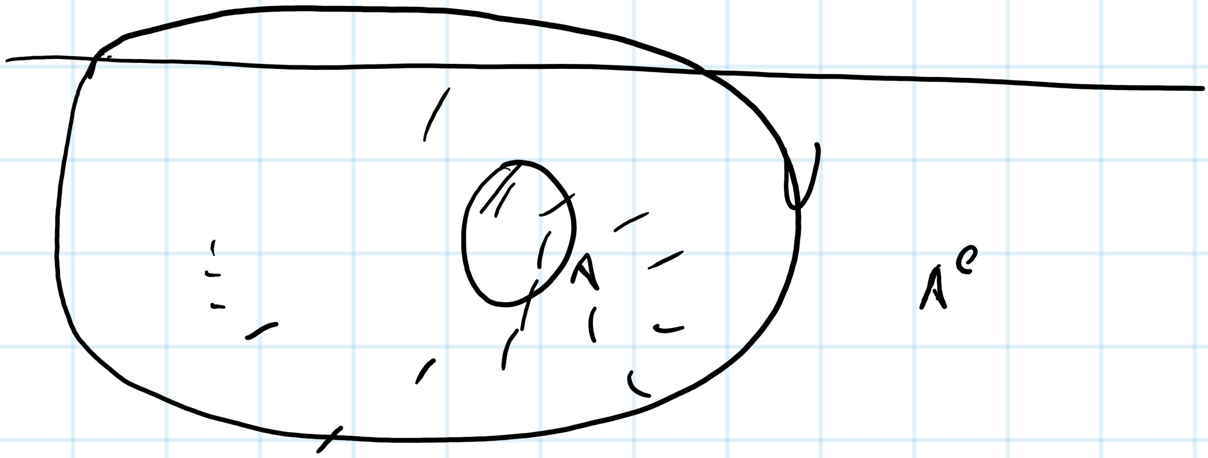
$$x \in A \cap (B \cup C) \Leftrightarrow \big(x \in A \text{ e } x \in (B \cup C) \big) \Leftrightarrow$$

$$\Leftrightarrow \big(x \in A \text{ e } ((x \in B) \text{ o } (x \in C)) \big) \Leftrightarrow$$

$$\iff \Big(\big((x \in A) \text{ e } (x \in B)\big) \text{ o } \big((x \in A) \text{ e } (x \in C)\big) \Big) \iff$$

$$\iff \left(x \in A \cap B \quad \text{o} \quad x \in A \cap C \right) \iff$$

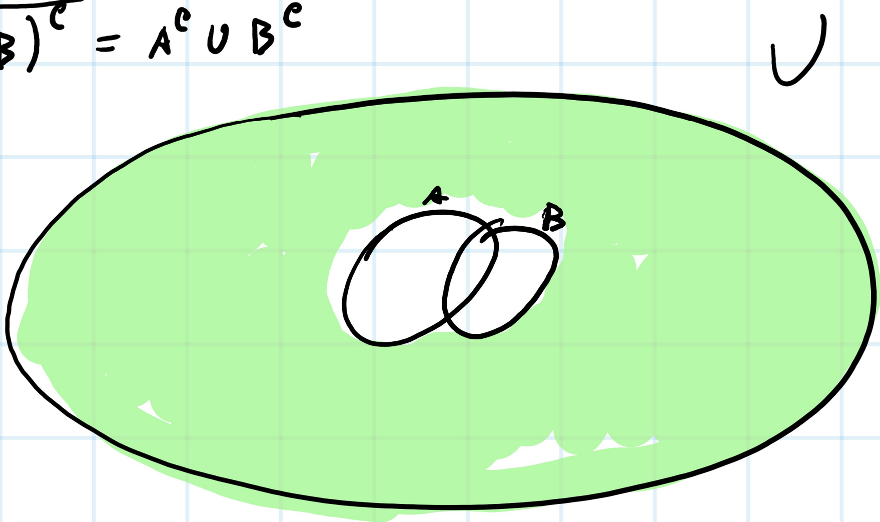
$$\iff x \in (A \cap B) \cup (A \cap C)$$

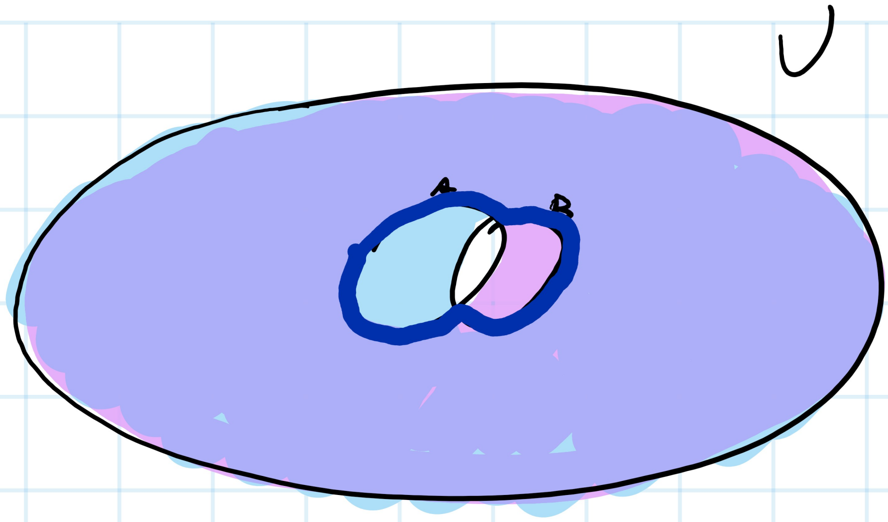


LEGGI DI DE MORGAN

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$





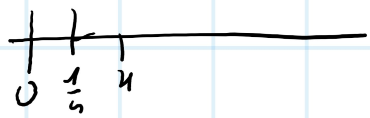
$\boxed{\text{DEF.}}$ DATI $\{A_i\}_{i \in I}$

$$\bigcap_{i \in I} A_i = \text{TUTTI E SOLI GLI } x \text{ t.c. } x \in A_i \ \forall i \in I$$

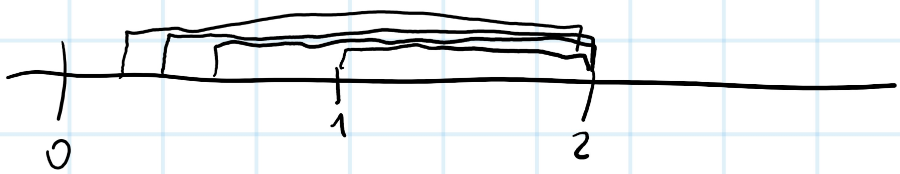
$$\bigcup_{i \in I} A_i = \text{TUTTI E SOLI GLI } x \text{ t.c. } \exists\, i \in I \text{ t.c. } x \in A_i$$

ESEMPIO

$$\bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 2\right] = (0, 2]$$



$$[1,2] \cup \left[\frac{1}{2}, 2\right] \cup \left[\frac{1}{3}, 2\right] \cup \dots \cup \left[\frac{1}{n}, 2\right] \cup \dots$$



RISPOSTA: NO PERCHÉ IL PRIMO NON CONTIENE 0 E IL SECONDO SI

ES. 4 DIRE SE SONO UGUALI I 2 INSIEMI:

$$\bigcup_{n\in\mathbb{N}}\left(\bigcap_{k\in\mathbb{N}}\left(\frac{1}{n+1}-\frac{1}{k+1},\frac{1}{n+1}+\frac{1}{k+1}\right)\right) \quad \text{E} \quad \bigcap_{k\in\mathbb{N}}\left(\bigcup_{n\in\mathbb{N}}\left(\frac{1}{n+1}-\frac{1}{k+1},\frac{1}{n+1}+\frac{1}{k+1}\right)\right)$$

$$I_{n,k} = \left(\frac{1}{n+1}-\frac{1}{k+1},\ \frac{1}{n+1}+\frac{1}{k+1}\right) =$$

$R = \frac{1}{n+1}$

$$I_{3,15} = \left(\frac{1}{4}-\frac{1}{15},\ \frac{1}{4}+\frac{1}{15}\right)$$

$$\left\{\frac{1}{n} \;\middle|\; n\in\mathbb{N}-\{0\}\right\}$$

$$\bigcup_{n\in\mathbb{N}}\left(\bigcap_{k\in\mathbb{N}}\left(\frac{1}{n+1}-\frac{1}{k+1},\ \frac{1}{n+1}+\frac{1}{k+1}\right)\right) = \bigcup_{n\in\mathbb{N}}\left(\left\{\frac{1}{n+1}\right\}\right) =$$

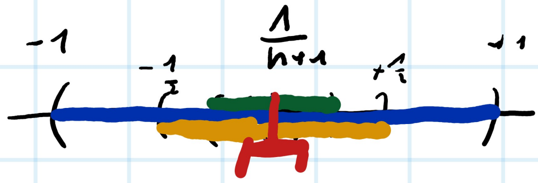
$$\left(\frac{1}{n+1}-1,\ \frac{1}{n+1}+1\right)$$

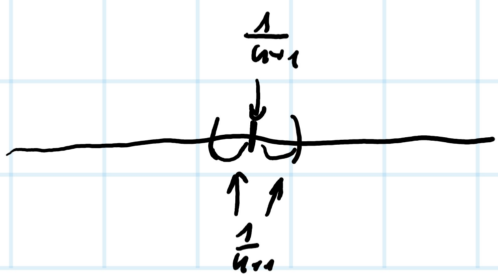
$$\left(\frac{1}{n+1}-\frac{1}{2},\ \frac{1}{n+1}+\frac{1}{2}\right)$$

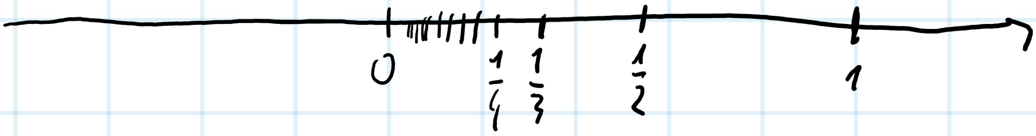
$\vdots$

$$\left(\frac{1}{n+1}-\frac{1}{k+1},\ \frac{1}{n+1}+\frac{1}{k+1}\right)$$

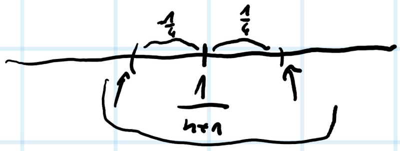
$\vdots$







$$0 \in \bigcap_{k \in \mathbb{N}} \left(\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1} - \frac{1}{k+1}, \frac{1}{n+1} + \frac{1}{k+1} \right) \right) =$$



$n > n$ CONTIENE 0