Lezione 3: Sottoinsiemi di R

INDICE

1) $\mathbb{Q}$
   - 1.1) DEF.
   - 1.2) CHIUSO PER SOMMA E PRODOTTO
   - 1.3) DENSO IN $\mathbb{R}$
   - 1.4) IN $\mathbb{R}$ ESISTE $\sqrt{2}$.
   - 1.5) $\sqrt{2} \notin \mathbb{Q}$
   - 1.7) $\mathbb{R}-\mathbb{Q}$ È DENSO IN $\mathbb{R}$
   - 1.8) $\mathbb{Q}$ NON È COMPLETO

2) ESERCIZI: 7 (S), 12 (S), 13 (V), 19, 17 (S, V)

P.A.

$$\left(m \cdot \frac{1}{n}\right)^2 = 2$$

WLOG POSSO SUPPORRE CHE m e n SIANO COPRIMI

$m^2 = 2n^2 \Rightarrow m$ PARI $\Rightarrow m = 2k$

$\Rightarrow (2k)^2 = 2n^2 \Rightarrow 4k^2 = 2n^2 \Rightarrow 2k^2 = n^2$

$\Rightarrow n$ PARI (ASS.)

$\boxed{\text{DEF.}}$ $\mathbb{Q}^+ = \{ n \cdot \frac{1}{m} \mid n, m \in \mathbb{N} \}$

$\mathbb{Q}^- = \{ q \mid -q \in \mathbb{Q}^+ \}$

$\mathbb{Q} = \{0\} \cup \mathbb{Q}^+ \cup \mathbb{Q}^-$

$\forall a,b,c \in \mathbb{Q} \subset \mathbb{R} \quad a+(b+c) = (a+b)+c$

$(m \cdot q)\left(\frac{1}{m} \cdot \frac{1}{q}\right) = \left(m \cdot \frac{1}{m}\right) \cdot \left(q \cdot \frac{1}{q}\right) =$

$= 1 \cdot 1 = 1$

$\boxed{\text{T. 1}}$ $\mathbb{Q}^+$ È STABILE PER $+$ e $\cdot$

$\boxed{\text{DIM}}$

$\boxed{q_1, q_2 \in \mathbb{Q}^+} \Rightarrow q_1 \cdot q_2 \in \mathbb{Q}^+$

$\Downarrow$

$q_1 = n \cdot \frac{1}{m}$

$q_2 = p \cdot \frac{1}{q}$

$q_1 \cdot q_2 = \left(n \cdot \frac{1}{m}\right) \cdot \left(p \cdot \frac{1}{q}\right) = (n \cdot p) \cdot \left(\frac{1}{m} \cdot \frac{1}{q}\right) = \underbrace{(n \cdot p) \cdot \frac{1}{m \cdot q}}_{\in \mathbb{Q}^+}$

$\Rightarrow q_1 + q_2 \in \mathbb{Q}^+$

$\left(n \cdot \frac{1}{m}\right) + \left(p \cdot \frac{1}{q}\right) = \left(n \cdot 1 \cdot \frac{1}{m}\right) + \left(p \cdot 1 \cdot \frac{1}{q}\right) =$

$= \left((n \cdot q)\left(\frac{1}{q} \cdot \frac{1}{m}\right)\right) + \left((p \cdot m)\left(\frac{1}{m} \cdot \frac{1}{q}\right)\right) =$

$= \left((n \cdot q) \cdot \frac{1}{q \cdot m}\right) + \left((p \cdot m) \cdot \frac{1}{q \cdot m}\right) =$

$= \underbrace{(n \cdot q + p \cdot m)}_{\in \mathbb{N}} \cdot \frac{1}{\underbrace{q \cdot m}_{\in \mathbb{N}}} \in \mathbb{Q}^+$

$\boxed{\text{PARENTESI}}$

$n, m \in \mathbb{N} \Rightarrow n \cdot m \in \mathbb{N}$

$\boxed{\text{DIM}}$

$\forall n \in \mathbb{N}$ SIA $A = \{ m \in \mathbb{N} \mid n \cdot m \in \mathbb{N} \}$

1) $1 \in A$ (ovvio)

2) $\underline{k} \in A \overset{?}{\Rightarrow} \underline{k+1} \in A$

$\overbrace{n \cdot k \in \mathbb{N}}$ $\quad n \cdot (k+1) = n \cdot k + n \cdot 1 = \boxed{\underset{\substack{\Downarrow \\ \mathbb{N}}}{n \cdot k} + \underset{\substack{\Downarrow \\ \mathbb{N}}}{n}}$ $\in \mathbb{N}$

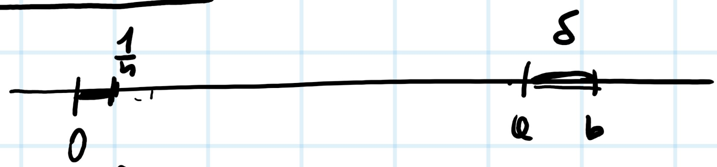
$\boxed{\text{T.}}$ $\mathbb{Q}$ È DENSO IN $\mathbb{R}$, CIOÈ $\forall a, b \in \mathbb{R}$, $\exists q \in \mathbb{Q}$ t.c. $a < q < b$

con $a < b$

$\boxed{\text{DIM}}$

FACCIO CASO $0 < a < b$

$\rightarrow \boxed{\forall \varepsilon > 0 \ \exists n \in \mathbb{N} \text{ t.c. } 0 < \frac{1}{n} < \varepsilon} \leftarrow$



PRENDO $\underline{\varepsilon} = \min(\underline{a}, b-a)$

PRENDO $n \in \mathbb{N}$ t.c. $0 < \frac{1}{n} < \varepsilon$

$A = \{ k \in \mathbb{N} \mid k \cdot \frac{1}{n} \leq a \}$

$A \neq \mathbb{N}$ PERCHÉ SE P.A. $A = \mathbb{N}$ ALLORA $\forall k \in \mathbb{N}$

$k \cdot \frac{1}{n} \leq a$ CIOÈ $k \leq a \cdot n$ (ASS.)

QUINDI $\exists k_0 \in \mathbb{N}$ t.c. $k_0 \notin A$

QUINDI A È FINITO.

QUINDI $\exists\ \bar{k} = \max\{A\}$

$\Downarrow$

$$\left.\begin{aligned} &\bar{k}\cdot\frac{1}{n} \le a \\ &(\bar{k}+1)\cdot\frac{1}{n} > a \end{aligned}\right\} \Rightarrow \boxed{(\bar{k}+1)\cdot\frac{1}{n}} < b$$

$\mathbb{Q}^+$

SE P.A. FOSSE $(\bar{k}+1)\cdot\frac{1}{n} \ge b$

$(\bar{k}+1)\cdot\frac{1}{n} + \left(-\frac{1}{n}\right) \ge b + \left(-\frac{1}{n}\right)$

$\left(\bar{k}\cdot\frac{1}{n}\right) \ge b + \left(-\frac{1}{n}\right) > a$

$\frac{1}{n} < b - a$

$a + \frac{1}{n} < b$

$a < b + \left(-\frac{1}{n}\right)$

$\boxed{\text{T.}}$ $\exists\, x \in \mathbb{R}$ t.c. $x^2 = 2$

$\boxed{\text{DIM}}$

$A = \{x \in \mathbb{R} \mid x > 0,\ x^2 < 2\}$ $\qquad 1 \in A$ (ovvio)

$B = \{x \in \mathbb{R} \mid x > 0,\ x^2 > 2\}$ $\qquad 2 \in \mathbb{R}$ (ovvio)

$\forall\, a \in A,\ b \in B \quad a^2 < 2 < b^2 \Rightarrow a^2 < b^2 \overset{?}{\Rightarrow} a < b$

$$a<b \Rightarrow 0<a^2<ab$$

$$\Rightarrow 0<ab<b^2$$

$$\Rightarrow \boxed{a^2<b^2}$$

$$\boxed{\begin{array}{l} a<b \Rightarrow a^2<b^2 \\ a=b \Rightarrow a^2=b^2 \\ a>b \Rightarrow a^2>b^2 \end{array}}$$

$$\boxed{a<b} \overset{?}{\Leftarrow} \boxed{a^2<b^2}$$

$a=b$
$a<b$
$a>b$

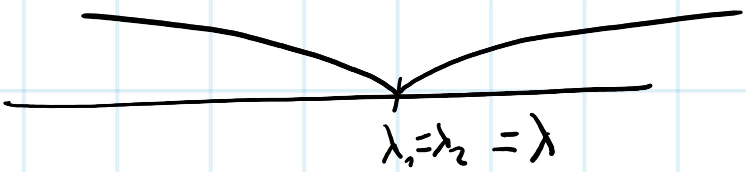
$\lambda_1 = \sup A$

$\lambda_2 = \inf B$

ESISTONO E $\boxed{\lambda_1 \le \lambda_2}$

$$\lambda_1 < \lambda_2$$

$$\lambda_1 < \alpha < \lambda_2$$



$$\lambda^2 = 2$$

MOSTRIAMO CHE $\boxed{\lambda^2 < 2}$ E $\boxed{\lambda^2 > 2}$ È ASSURDO

$$\exists\, \varepsilon > 0 \text{ t.c. } \underbrace{(\lambda + \varepsilon)^2 < 2}$$

$$\lambda^2 + 2\lambda\varepsilon + \varepsilon^2 < 2 \quad ?$$

$$\varepsilon(2\lambda + \varepsilon) < \boxed{2 - \lambda^2}$$

PRENDO ε t.c. $\begin{cases} \varepsilon > 0 \\ \varepsilon < 2\lambda \\ \varepsilon < \dfrac{2-\lambda^2}{4\lambda} \end{cases}$ $\quad \min\{\circ, \circ\}$

0

$$\boxed{\varepsilon \cdot (2\lambda + \varepsilon)} < \varepsilon \cdot (2\lambda + 2\lambda) = \varepsilon \cdot 4\lambda < \frac{2-\lambda^2}{\cancel{4\lambda}} \cancel{4\lambda} = 2 - \lambda^2$$

QUINDI $\exists\, \varepsilon > 0$ t.c.

$$\varepsilon(2\lambda + \varepsilon) < 2 - \lambda^2$$

CIOÈ

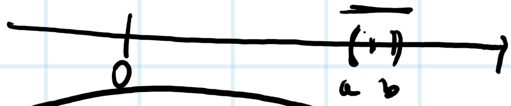
$$\lambda^2 + \varepsilon(2\lambda + \varepsilon) < 2$$

CIOÈ $\quad (\lambda + \varepsilon)^2 < 2 \quad$ (ASS. PERCHÈ CONTRADDICE $\lambda = \sup A$)

[. . .]

QUINDI $\lambda^2 = 2$

$\boxed{\text{T.}}$ $\mathbb{R}-\mathbb{Q}$ È DENSO IN $\mathbb{R}$



$\boxed{\text{DIMO}}$

1) $q \in \mathbb{Q}$ $\quad q\cdot\sqrt{2} \notin \mathbb{Q}$

p.a. $q\cdot\sqrt{2} = p$ con $p \in \mathbb{Q}$

$\sqrt{2} = p\cdot q^{-1} \in \mathbb{Q}$

2) $\forall \varepsilon > 0 \ \exists n \in \mathbb{N}$ t.c. $\quad 0 < \sqrt{2}\cdot\frac{1}{n} < \varepsilon$

$0 < \frac{1}{n} < \varepsilon \cdot \frac{1}{\sqrt{2}}$

3) $\varepsilon = \min(b-a,\ a)$

n t.c. $\boxed{\sqrt{2}\cdot\frac{1}{n} < \varepsilon}$ $\quad$ [.....]

m t.c. $m\cdot\left(\sqrt{2}\cdot\frac{1}{n}\right) \in (a,b)$ $\quad$ [. . .]

T. $\mathbb{Q}$ NON È COMPLETO

$\boxed{\text{DIM}}$ (idea)

$A = \{q \in \mathbb{Q}^+ \mid q^2 < 2\}$

$B = \{q \in \mathbb{Q}^+ \mid q^2 > 2\}$

[...] $\quad \lambda = \sup(A) = \inf(B)$ $\quad$ (in $\mathbb{R}$)

$\notin$
$\mathbb{Q}$

$\{0, 1, 2, \ldots, n, \ldots\}$

PAG. 19

19) $A = \left\{ \frac{n+1}{m+1} + \frac{m+1}{n+1} \;\middle|\; n, m \in \mathbb{N} \right\}$

$x \qquad \frac{1}{x}$

$B = \left\{ \bigcirc \;\middle|\; n \in \mathbb{N}, m = 0 \right\} \subset A$

$= \left\{ \frac{n+1}{1} + \frac{1}{n+1} \;\middle|\; n \in \mathbb{N} \right\} =$

$n+1+\frac{1}{n+1} > \boxed{n}$

$\leq x$

NO

$SUP(A) = +\infty$

$MAX(A)$ NON ESISTE

$INF(A) = 2$

$MIN(A) = 2$?

$2 \in A \quad (n = m)$

$\frac{n+1}{m+1} + \frac{m+1}{n+1}$

$x + \frac{1}{x} = (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 = (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2 \cdot + 2 =$

$$= \left(\sqrt{x}\right)^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2 \cdot \sqrt{x} \cdot \frac{1}{\sqrt{x}} + 2 =$$

$$= \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2 \geq 2$$

$$A = \left\{ \frac{2m+1}{2n+2} + \frac{2n+2}{2m+1} \;\middle|\; n, m \in \mathbb{N} \right\}$$

$$\frac{1}{2n+2} + \frac{2n+2}{}$$

SUP(A) = +∞

MAX(A) = NON ESISTE

INF(A) = 2

MIN(A) NO

$$\frac{2n+1}{2n+2} + \frac{2n+2}{2n+1} = \frac{2n+2-1}{2n+2} + \frac{2n+1+1}{2n+1} =$$

$$= 1 - \frac{1}{2n+2} + 1 + \frac{1}{2n+1} =$$

$$= 2 + \frac{\cancel{-2n}-1+\cancel{2n}+2}{(2n+2)(2n+1)} = 2 + \frac{1}{(2n+2)(2n+1)}$$