Lezione 5: Successioni a valori in $\mathbb{R}$

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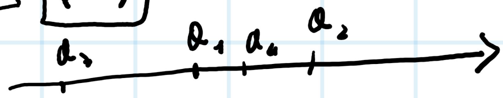
PER CASA: ESERCIZI 21-33 PAG. 65 ESERCIZIARIO

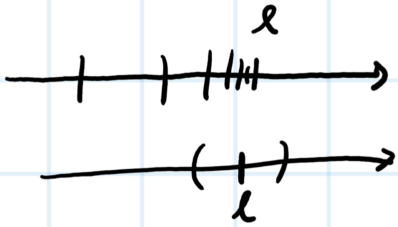
DEF. 1 DIREMO $\overbrace{\text{SUCC. A VALORI IN } \mathbb{R}}$ $(a_n)_{\boxed{n\in\mathbb{N}}}$ $\boxed{(a_n)}$



UNA FUNZIONE $a : \mathbb{N} \to \mathbb{R}$

$n \longmapsto a_n$

DEF. 2 DATA (a_n) E DATO $\ell \in \mathbb{R}$



DIREMO CHE "$a_n \to \ell$" OPPURE "$\lim_{n\to+\infty} a_n = \ell$"

SE:

$$\forall \varepsilon > 0 \ \underbrace{\exists n_0 \in \mathbb{N} \ \text{t.c.} \ \forall n \geq n_0}_{\text{(DEFINITIVAMENTE IN } n)} \ \text{SI HA} \ \boxed{a_n \in I_\ell(\varepsilon)}$$

$\ell - \varepsilon < a_n < \ell + \varepsilon$

$|a_n - \ell| < \varepsilon$

$\boxed{\text{FRASI E QUANTIFICATORI}}$

3 PARI falsa

5 DISPARI VERA

$P(n) =$ "n PARI"

$\underline{\forall n \in \mathbb{N} \ P(n)}$ F

$\exists n \in \mathbb{N} \ P(n)$ V

DEF. IN n $P(n)$ F

$\boxed{\text{FREQ}}$ IN n $P(n)$ V

$\boxed{\forall n \in \mathbb{N} \ \exists k \geq n \ \text{t.c.} \ P(k)}$

"$\neg\left(\forall n \in \mathbb{N} \ P(n)\right)$" $= \exists n \in \mathbb{N} \ \text{t.c.} \ \neg(P(n))$

"$\neg\left(\exists n \in \mathbb{N} \ P(n)\right)$" $= \forall n \in \mathbb{N} \ \neg(P(n))$

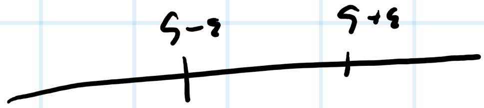
$P(n) :=$ " $a_n \in (\ell-\varepsilon, \ell+\varepsilon)$ "

" $\neg$(DEF IN n P(n)) " $\Leftrightarrow$ " FREQ IN n $\neg$(P(n)) "

$\boxed{\text{ES.}}$ $\neg(a_n \to 5)$

$\neg(\forall \varepsilon>0$, DEF. IN n, $|a_n-\ell|<\varepsilon)$

" $\exists \varepsilon>0$ t.c. FREQ IN n $|a_n-\ell|\geq\varepsilon$ "



$\boxed{\text{ES.1}}$ $\frac{1}{n} \to 0$

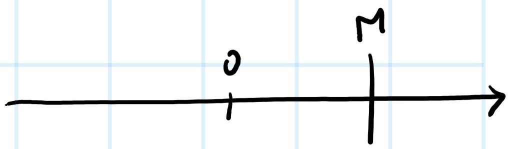
$\forall \varepsilon>0$, DEF IN n, $\left|\frac{1}{n}\right|<\varepsilon$

$\forall \varepsilon>0\ \exists n_0 \in \mathbb{N}$ t.c. $\boxed{0<\frac{1}{n_0}<\varepsilon}$

$\forall n \geq n_0$ $\quad 0<\frac{1}{n}<\varepsilon$ $\qquad \forall n>n_0$ $\boxed{0<\frac{1}{n}<\frac{1}{n_0}}$

$\boxed{\text{DEF.2}}$ DATA (a_n) DIREMO CHE $a_n \to +\infty$ ($-\infty$) SE

$\forall M>0$ (<0) DEF. IN n $a_n>M$ ($<M$)



$\boxed{\text{ES.2}}$ " $\lim_{n\to+\infty} n = +\infty$ " (a_n)

$\forall M>0$ DEF IN n $n>M$

$\boxed{ES.3}$ $\dfrac{2n^8+2n+5}{n^8+n+3} \to 2$ PER $n \to +\infty$

$\forall \varepsilon > 0$ DEF IN n $\left| \dfrac{2n^8+2n+5}{n^8+n+3} - 2 \right| < \varepsilon$

$$\left| \frac{-1}{n^8+n+3} \right| < \varepsilon$$

$\boxed{n^8+n+3>0}$

$n^2+n+3>0$

$$\frac{|-1|}{|n^8+n+3|} < \varepsilon$$

$$\frac{1}{n^8+n+3} < \varepsilon$$

perché membri positivi

$$\boxed{n^8+n+3} > \boxed{\frac{1}{\varepsilon}}$$

$$n^8+n+3 > n > \frac{1}{\varepsilon}$$

$\boxed{T.1}$ $\boxed{\text{UNICITÀ DEL LIMITE}}$

DATA (a_n) SE $a_n \to \ell$ ALLORA NON TENDE A NIENTE ALTRO.

$\boxed{DIM}$

"$a_n \to \ell$" PRESO $\ell_2 \neq \ell$

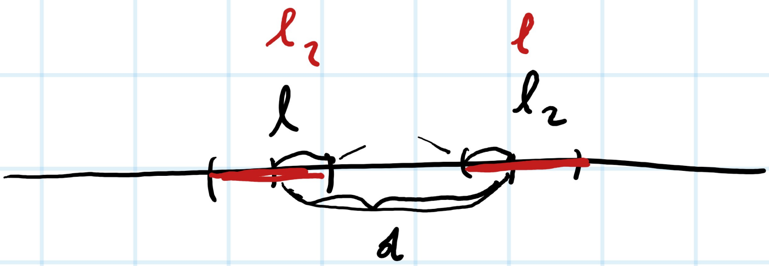
$\forall \varepsilon > 0$ DEF IN n $|a_n - \ell| < \varepsilon$

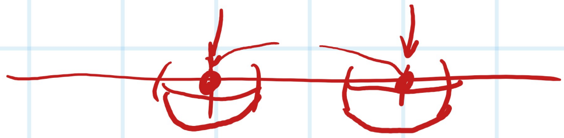
$$\varepsilon = \frac{\overbrace{|\ell - \ell_2|}^{d}}{3} = \frac{d}{3}$$

DEF IN n $|a_n - l| < \frac{d}{3} \iff l - \frac{d}{3} < a_n < l + \frac{d}{3}$

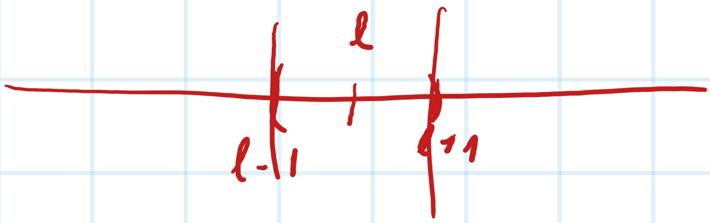
SE P.A. $a_n \to l_2$

DEF IN n $|a_n - l_2| < \frac{d}{3} \iff l_2 - \frac{d}{3} < a_n < l_2 + \frac{d}{3}$





$a_n \to l$

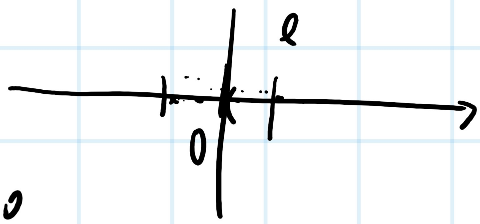


$\boxed{\text{T.}}$ DATI (a_n) E l t.c. $a_n \to l$

ALLORA

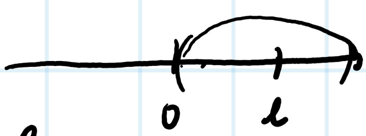
→ 1) SE $l > 0$ DEF. IN n ANCHE $a_n > 0$

2) $<$ $<$



$\boxed{\text{DIM.}}$ $a_n \to l \iff \forall \varepsilon > 0$ DEF. IN n $|a_n - l| < \varepsilon$

$l - \varepsilon < a_n < l + \varepsilon$



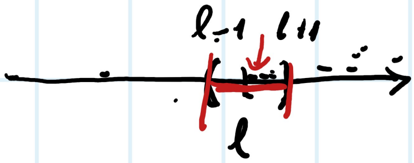
$\boxed{\varepsilon = l}$ DEF. IN n $0 < a_n < 2l$

T. (LIMITATEZZA DI (a_n) CONVERGENTI)

SE (a_n) HA LIM. FINITO ALLOR (a_n) È LIMITATA.

$\boxed{\text{DIM}}$

SE $a_n \to \ell \in \mathbb{R}$ ALLORA



$\forall \varepsilon > 0$ DEF. IN n $\ell - \varepsilon < a_n < \ell + \varepsilon$

$\varepsilon = 1$ $\exists n_0 \in \mathbb{N}$ t.c. $\forall n \geq n_0$ $\ell - 1 < a_n < \ell + 1$

$A = \{a_1, a_2, \dots, a_{n_0 - 1}\}$ $m = \text{MIN}(A)$ $(\ell - 1, \ell + 1)$

$M = \text{MAX}(A)$ $[m, M]$

$a = \text{MIN}(m, \ell - 1)$

$b = \text{MAX}(M, \ell + 1)$

$\forall n \in \mathbb{N}$ $a_n \in [a, b]$

$\boxed{\text{DEF.}}$ DATA (a_n), DIREMO CHE

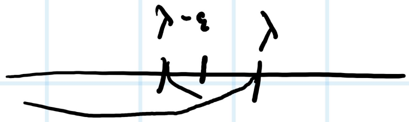
1) a_n È (STRETT.) CRESCENTE SE $\forall n \in \mathbb{N}$ $a_n \leq a_{n+1}$ ($<$)

(ST) DECRESC... ($\geq$, $>$)

T. DATA (a_n), SE (a_n) È CRESCENTE ALLORA $\lim_{n\to+\infty} a_n$ ESISTE SEMPRE E SI HA

$$\boxed{\lim_{n\to+\infty} a_n = \sup\{a_n \mid n\in\mathbb{N}\}}$$

DIM



I° (a_n) limitata

$\lambda = \sup\underbrace{\{a_n \mid n\in\mathbb{N}\}}_{A} \in \mathbb{R}$

$\to \forall x\in A \quad \lambda \ge x$

$\forall \varepsilon>0 \quad \lambda-\varepsilon$ NON È MAGGIORANTE $\to \forall\varepsilon>0 \ \exists x\in A$ t.c. $\lambda-\varepsilon<x$

$$\boxed{\begin{array}{l} a_n \le \lambda \quad \forall n \\ \exists n_0\in\mathbb{N} \text{ t.c. } a_{n_0} > \lambda-\varepsilon \end{array}}$$

$\forall\varepsilon>0 \quad \exists n_0\in\mathbb{N}$ t.c. $\lambda-\varepsilon<a_{n_0}\le\lambda$

(a_n) È CRESC. $\Rightarrow \forall n\ge n_0 \quad \lambda-\varepsilon<a_{n_0}\le a_n\le\lambda$

"$\forall\varepsilon>0$ DEF. IN n $\lambda-\varepsilon<a_n\le\lambda<\lambda+\varepsilon$" $\Leftrightarrow a_n\to\lambda$

II° (a_n) NON LIMITATA $\overset{\text{CRE}}{\Rightarrow}$ $\underbrace{\sup\{a_n \mid n\in\mathbb{N}\}=+\infty}$

$\forall M>0 \quad \exists n_0\in\mathbb{N}$ t.c. $a_{n_0}>M$

MA POICHE (a_n) È CRESC. $\quad n\ge n_0 \quad a_n\ge a_{n_0}>M$

"$\forall M>0$ DEF. IN n $a_n>M$" $a_n\to+\infty$

T1 CONF DATE (a_n), (b_n), (c_n) t.c. DEF. IN n SI ABBIA

$$a_n \le b_n \le c_n$$

SE $\exists\, l \in \mathbb{R}$ t.c. $a_n \to l$ $c_n \to l$ ALLORA ANCHE $b_n \to l$

T2 CONF DATE (a_n), (b_n) t.c. DEF IN n $a_n \le b_n$ ALLORA

1) SE $a_n \to +\infty$ ALLORA ANCHE $b_n \to +\infty$

2) SE $b_n \to -\infty$ $a_n \to -\infty$

DIM 1 DEF. IN n $\boxed{a_n \le b_n \le c_n}$

$\forall \varepsilon > 0$ DEF. IN n $l - \varepsilon < a_n < l + \varepsilon$

$\forall \varepsilon > 0$ DEF. IN n $l - \varepsilon < c_n < l + \varepsilon$

$\forall \varepsilon > 0$ DEF IN n $l - \varepsilon < a_n \le b_n \le c_n < l + \varepsilon$

$\boxed{\forall \varepsilon > 0 \text{ DEF IN } n \quad l - \varepsilon < b_n < l + \varepsilon}$ $b_n \to l$

DIM 2 $\boxed{a_n \to +\infty \quad \text{DEF IN } n \quad b_n \ge a_n}$ $\Rightarrow$ $b_n \to +\infty$

$\forall M > 0$ DEF IN n $a_n > M$ $\forall M > 0$ DEF. IN n $b_n > M$

$\frac{100}{n^2} \to 0$?

$\forall n$ DEF. IN n

$0 \leq \frac{100}{n} \cdot \frac{1}{n} \leq \frac{1}{n}$

$$\begin{array}{ccccc} 0 & \leq & \frac{100}{n^2} & \leq & \frac{1}{n} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

$n^7 - n^3 \to +\infty$

$n \geq 2$

$\boxed{n^7 - n^3} = n^3(n^4 - 1) \geq n^3 \cdot 1 = n^3 > \boxed{n}$

DEF. IN n $\quad \underbrace{n^7 - n^3}_{\downarrow \; +\infty} > \underset{\downarrow \; +\infty}{n}$

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