Lezione 6: Operazioni sui limiti

<u>INDICE</u>

ESEMPI: TROVARE LIMITE DI

$A_n = \frac{5n^2 - 2n + 3}{3n^2 - 1}$

$A_n = n + \sin n$

$A_n = \frac{1}{n} \cdot \sin(n)$

$A_n = n \cdot \left(\frac{11}{10} - \sin(n) \right)$

$A_n = \sin(\sin n) / \sqrt{n}$

$A_n = 2^n / (\cos^4 n + \sin^6 n)$

$A_n = \arctan n / \left(\sqrt{n^2 + \sin n} - \sqrt{n^2 + \cos n} \right)$

$A_n = \frac{\sin \frac{1}{n}}{(-1)^n + \sin \frac{1}{n}}$

$\boxed{T.1}$ DATE (a_n) (b_n) T.C. $a_n \to l \in \mathbb{R}$ $\quad b_n \to L \in \mathbb{R}$

ALLORA SI HA:

$1 \to 1)$ $a_n + b_n \to l + L$

$1 \to 2)$ $a_n \cdot b_n \to l \cdot L$

$1 \to 3)$ SE $L \neq 0$ $\quad \dfrac{a_n}{b_n} \to \dfrac{l}{L}$

DISUGUAGLIANZA TRIANGOLARE: $|x+y| \leq |x| + |y|$

$\boxed{\text{DIM}}$ $\boxed{\text{I° CASO}}$ $x > 0,\ y > 0$

$|x| + |y| = x + y = |x+y|$

$\boxed{\text{II° CASO}}$ $\boxed{x < 0\ y < 0}$

$|x| + |y| = -x - y = -(x+y) = |x+y|$

$\boxed{\text{III CASO}}$ $\boxed{x > 0,\ y < 0}$

SE $x+y > 0$: $|x+y| = x + y = |x| - |y| \geq |x| \geq |x| + |y|$

SE $x+y < 0$: $|x+y| = -x - y = -|x| + |y| \geq |y| \geq |x| + |y|$

$\boxed{\text{DIM}}$ $\underline{\forall \varepsilon > 0}$ DEF. IN n $\quad \boxed{|(a_n + b_n) - (l + L)| < \varepsilon}$ $\quad (?)$

$$\begin{cases} \forall \varepsilon > 0 \text{ DEF. IN } n. & |a_n - l| < \dfrac{\varepsilon}{2} \\ \forall \varepsilon > 0 \text{ DEF. IN } n & |b_n - L| < \dfrac{\varepsilon}{2} \end{cases}$$

$$|(a_n + b_n) - (l + L)| = |(a_n - l) + (b_n - L)| \leq |a_n - l| + |b_n - L| \leq$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

2) $\underline{\forall \varepsilon > 0}$ DEF. IN n $\quad \boxed{|a_n \cdot b_n - l \cdot L| < \varepsilon}$ $\quad (?)$

$\boxed{\text{DEF. IN } n}$ $|a_n - l| < \dfrac{\varepsilon}{2|L|}$

SIA $M > 0$ T.C. $|a_n| < M$

$\boxed{\text{DEF. IN } n}$ $|b_n - L| < \dfrac{\varepsilon}{2M}$

$$|a_n \cdot b_n - l \cdot L| = |a_n \cdot b_n - a_n \cdot L + a_n \cdot L - l \cdot L| =$$

$$= |a_n(b_n - L) + L(a_n - l)| \leq$$

$$\leq |a_n(b_n - L)| + |L(a_n - l)| = |a_n| \cdot |b_n - L| + |L| \cdot |a_n - l| \leq$$

$$\leq \cancel{M} \cdot \frac{\varepsilon}{2\cancel{M}} + \cancel{|L|} \cdot \frac{\varepsilon}{2\cancel{|L|}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$L = 0$

$|a_n| |b_n| \leq M \cdot \dfrac{\varepsilon}{M} = \varepsilon$

(*)

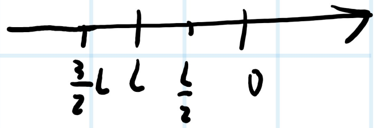
3) $\frac{a_n}{b_n} \to \frac{l}{L}$

$\boxed{L \neq 0}$ NOTA: PER PERM. SEGNO ANCHE $b_n \neq 0$ DEF. IN n, QUINDI (*) HA SENSO

$a_n \cdot \frac{1}{b_n} \to l \cdot \frac{1}{L}$

BASTA MOSTRARE CHE:

$\frac{1}{b_n} \to \frac{1}{L}$



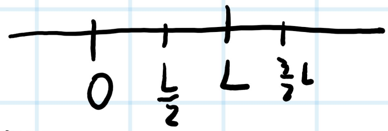
$\frac{L}{2} < b_n < \frac{3}{2}L$

$|b_n - L| < \frac{L}{2}$

$\forall \varepsilon > 0$ DEF. IN n $\left|\frac{1}{b_n} - \frac{1}{L}\right| < \varepsilon$

$\left|\frac{L - b_n}{b_n \cdot L}\right| = \frac{1}{|b_n|} \cdot \frac{1}{|L|} \cdot |b_n - L| \leq$

$\leq \frac{2}{|L|} \cdot \frac{1}{|L|} \cdot \frac{\varepsilon \cdot |L|^2}{2} = \varepsilon$



$\boxed{L>0}$

$\frac{L}{2} < b_n < \frac{3}{2}L$ $\boxed{0 < \frac{1}{b_n} < \frac{2}{L}}$

$|b_n - L| < \frac{\varepsilon}{\frac{2}{|L|^2}} = \frac{\varepsilon \cdot |L|^2}{2}$

$\boxed{L<0}$

$\frac{3}{2}L < b_n < \frac{L}{2}$

$\boxed{0 < \frac{1}{-b_n} < \frac{2}{|L|}}$

$0 < \frac{1}{|b_n|} < \frac{2}{|L|}$

$\boxed{T.}$ DATE (a_n), (b_n) t.c. $a_n \to +\infty$ $(-\infty)$ E (b_n) INF. LIM. (SUP)

$\exists \boxed{\lambda} \in \mathbb{R}$ MINORANTE DI $\{b_n\}_{n \in \mathbb{N}}$

ALLORA $a_n + b_n \to +\infty$ $(-\infty)$

$\boxed{\text{DIM}}$

$\forall M \in \mathbb{R}$ DEF. IN n $a_n + b_n > M$

$M - \lambda$

Def in n $a_n > M - \lambda \implies a_n + b_n > (M - \lambda) + \lambda = M$

GEN 1 PRODOTTO

$\boxed{T.}$ DATE (a_n) (b_n) T.C $a_n \to +\infty$ b_n t.c. $\exists \lambda > 0$ t.c. DEF IN n $b_n > \lambda$

ALLORA $a_n \cdot b_n \to +\infty$

$\boxed{DIM.}$

$\forall M > 0$ DEF IN n $a_n \cdot \boxed{b_n} > M$

$$a_n \cdot b_n > \frac{M}{\lambda} \cdot \lambda = M$$

DEF. IN n $a_n > \frac{M}{\lambda}$

$\forall \varepsilon > 0$ DEF IN n $|a_n - 0| < \varepsilon$

$\cdot \; \cdot \; - \; - \; \cdot \; - \; \big| |a_n| - 0 \big| < \varepsilon$

$|a_n| < \varepsilon$

$\boxed{T.}$ GEN. 2 PRODOTTO

DATE (a_n) (b_n) T.C. $\underline{a_n \to 0}$ E $\underbrace{b_n \text{ LIMITATA}}$

$\exists k > 0$ t.c. $|b_n| < k$

ALLORA $a_n \cdot b_n \to 0$

$\alpha \cdot a_n \to \alpha l$

$\boxed{DIM}$

$$0 \le |a_n \cdot b_n| \le |a_n| \cdot K$$

$$\underbrace{-K \cdot |a_n|}_{\downarrow \, 0} < \underbrace{\boxed{a_n \cdot b_n}}_{\downarrow \, 0} < \underbrace{K \cdot |a_n|}_{\downarrow \, 0}$$

$a_n \to l$

$b_n \to L$

$\alpha a_n + \beta b_n \to \alpha l + \beta L$

T. (GEN. QUOZIENTE) DATE (a_n) E (b_n) t.c. A TERMINI POSITIVI

$b_n \to 0$ E $\boxed{\exists k > 0 \text{ DEF IN n } a_n > k >}$

ALLORA $\frac{a_n}{b_n} \to +\infty$

$\boxed{DIM}$ $\forall M > 0$ DEF. IN n $\frac{a_n}{b_n} > \underline{M}$ (?)

$a_n > K$

$b_n \to 0$

$\Downarrow$

$\forall \varepsilon > 0$ DEF IN n $b_n < \varepsilon$

$$\frac{a_n}{b_n} > K \cdot \frac{1}{b_n} > K \cdot \frac{M}{K} = M$$

$\frac{1}{b_n} \uparrow \frac{M}{K}$

$b_n < \frac{K}{M}$ DEF IN n

a_n, b_n T. POSITIVA

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(2) ($b_n \to +\infty$ E a_n LIMITATA) ALLORA $\frac{a_n}{b_n} \to 0$

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(3) $a_n \to 0$ E (b_n t.c. $\exists c > 0$ t.c. DEF. IN n $b_n > c$) ALLORA $\frac{a_n}{b_n} \to 0$

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(4) ($a_n \to +\infty$ E b_n limitata) ALLORA $\left(\frac{a_n}{b_n} \to +\infty\right)$

$\boxed{\text{DIMO PER CASA}}$

ESEMPI

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$A_n = \sin(\sin n) / \sqrt{n}$

$A_n = 2^n / (\cos^4 n + \sin^4 n)$

$A_n = \arctan n / \left(\sqrt{n^2 + \sin n} - \sqrt{n^2 + \cos n} \right)$

$A_n = \frac{\sin \frac{1}{n}}{(-1)^n + \sin \frac{1}{n}}$

$\frac{1}{n} \to 0$

$n \to +\infty$

$$\lim_{n \to +\infty} \frac{5n^2 - 2n + 3}{3n^2 - 1} = \lim_{n \to +\infty} \frac{5 - 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} \cdot \frac{1}{n}}{3 - \frac{1}{n} \cdot \frac{1}{n}} = \frac{5 - 0 - 0}{3 - 0} = \frac{5}{3}$$

$$\lim_{n \to +\infty} \underset{\uparrow}{n} + \underset{\uparrow}{\sin n} = +\infty$$

$$\lim_{n \to +\infty} \frac{\sin n}{n} = \lim_{n \to +\infty} \underbrace{\boxed{\sin n}}_{\text{LIMIT.}} \cdot \underbrace{\boxed{\frac{1}{n}}}_{\downarrow \, 0} = 0$$

$$\lim_{n \to +\infty} \overset{+\infty}{\overset{\uparrow}{n}} \cdot \underbrace{\left(\frac{11}{10} - \cos n \right)}_{\geq \frac{1}{10}} = +\infty$$

$$\lim_{n \to +\infty} \frac{\sin(\sin n)}{\sqrt{n}} = 0$$

$$\underset{\downarrow \, 0}{-\frac{1}{\sqrt{n}}} < \underbrace{\frac{\sin(\sin n)}{\sqrt{n}}}_{\downarrow \, 0} < \underset{\downarrow \, 0}{\frac{1}{\sqrt{n}}}$$

$\sqrt{n} \to \ell$

$\sqrt{n} \cdot \sqrt{n} \to \ell^2$ $\quad n \to \ell^2$

$n \to +\infty$

$\sqrt{n} \to +\infty$