Analisi Matematica 1 - 6 Ottobre 2023 - docente: Callegari

Lezione 7: Infiniti e infinitesimi

INDICE

0) ALCUNE FUNZIONI (PER POTER FARE ESEMPI ED ESERCIZI)

- 0.1) POLINOMI ($a_n \to \ell \Rightarrow P(a_n) \to P(\ell)$)
- 0.2) RADICE (CRESCENTE, BIUNIVOCA SU $\mathbb{R}^+$, $a_n \to \ell \Rightarrow \sqrt{a_n} \to \sqrt{\ell}$) (ANCHE $\sqrt[k]{\dots}$)
- 0.3) FUNZIONE ESPONENZIALE: (SU $\mathbb{N}$) $\to$ (SU $\mathbb{Z}$) $\to$ [(SU $\mathbb{Q}$) MONOTONIA E DENSITÀ IMMAGINE] $\to$ [(SU $\mathbb{R}$), $a_n \to \ell \Rightarrow A^{a_n} \to A^{\ell}$]
- 0.4) FUNZIONE LOGARITMO

1) DEF. DI o-PICCOLO, "STESSO ORDINE", ASINT. EQUIV., O-GRANDE.

2) ESEMPI: $\frac{1}{n^2} = o\left(\frac{1}{n}\right)$, $\sqrt{n^2+1} \asymp n$, $n + \frac{1+(-1)^n}{2} n^2 = O(n^2)$

3) CATENA INFINITI

4) ESEMPI:

$$
\begin{array}{llll}
4.1) & a_n = n^{10} & b_n = 2^n & c_n = \sqrt{3^n} \\
4.2) & a_n = 2^{n^2} & b_n = 16^n & c_n = n^n \\
4.3) & a_n = \frac{2^n}{n^{100}} & b_n = \sqrt{2^n} & c_n = 4^{\sqrt{n}} \\
4.4) & a_n = \left(2 + \frac{1}{n}\right)^n & b_n = \left(3 - \frac{1}{n}\right)^n & c_n = (\log_2 n)^{\sqrt{n}} \\
4.5) & a_n = \log_2(n!) & b_n = \sqrt{1+n^2} & c_n = \log_2(1+2^n) \\
4.6) & a_n = (n^2)! & b_n = (n^2)^n & c_n = (n!)^2
\end{array}
\Bigg\} \text{ METTERE IN ORDINE DI INFINITO}
$$

$\boxed{a_n \to \ell}$

$\boxed{f(a_n) \to f(\ell)}$

$$p(x) = A_k x^k + A_{k-1} x^{k-1} + \dots + A_1 x + A_0$$

$\boxed{a_n \to \ell}$

$$p(a_n) = \underline{A_k (a_n)^k} + \underline{A_{k-1}(a_n)^{k-1}} + \dots + A_1 a_n + A_0 \to$$

$$\to A_k \ell^k + A_{k-1} \ell^{k-1} + \dots + A_1 \ell + A_0 = p(\ell)$$

DATO $x \geq 0$ $\exists!\, y \geq 0$ t.c. $y^2 = x$

$\boxed{x \longmapsto \sqrt{x}}$ $\boxed{\begin{array}{l} a_n \to \ell \\ \sqrt{a_n} \to \sqrt{\ell} \end{array}}$

$\boxed{\text{T.}}$ SE (a_n) È A VAL. POSITIVI E $a_n \to \ell > 0$ ALLORA $\sqrt{a_n} \to \sqrt{\ell}$

$\boxed{\text{DIM}}$

(S) $\boxed{\forall \varepsilon > 0 \text{ DEF IN } n \quad |\underline{a_n - \ell}| < \varepsilon\sqrt{\ell}}$

$\forall \varepsilon > 0$ DEF IN n $\underline{|\sqrt{a_n} - \sqrt{\ell}|} < \varepsilon$ (?)

$$|\sqrt{a_n} - \ell| = \frac{|\sqrt{a_n} - \sqrt{\ell}|}{1} = \frac{|a_n - \ell|}{\sqrt{a_n} + \sqrt{\ell}} \leq \frac{|a_n - \ell|}{\sqrt{\ell}} \underset{\text{DEF IN } n}{<} \frac{\varepsilon\sqrt{\ell}}{\sqrt{\ell}} = \varepsilon$$

$$(a^k - b^k) = (a-b)\underline{(a^{k-1} + a^{k-2} b + a^{k-3} b^2 + \dots + a b^{k-2} + b^{k-1})}$$

$$\left|\frac{\sqrt[k]{a_n}-\sqrt[k]{l}}{1}\right| \le \frac{|a_n - l|}{(\sqrt[k]{l})^{k-1}} \le$$

$$\left(\left(\sqrt[k]{a_n}\right)^{k-1} + \left(\sqrt[k]{a_n}\right)^{k-1}\cdot\sqrt[k]{l} + \dots\dots \boxed{/\!/}\right)$$

$$a^n = \underbrace{\overbrace{a\cdot a \cdots a}^{n}}$$

$$\boxed{a>1}$$

$$a^n \cdot a^m = a^{n+m}$$

$$a^{\frac{n}{m}} = \sqrt[m]{a^n}$$

$$a^{-\frac{n}{m}} = \frac{1}{a^{\frac{n}{m}}}$$

$$\underbrace{\frac{n}{m} > \frac{p}{q}} \overset{?}{\Rightarrow} a^{\frac{n}{m}} > a^{\frac{p}{q}}$$

$$\Updownarrow$$

$$nq > mp$$

$$\left(a^{\frac{n}{m}}\right)^{mq} > \left(a^{\frac{p}{q}}\right)^{mq}$$

$$\left(\left(\sqrt[m]{a^n}\right)^{mq}\right) = \boxed{a^{nq}} > \boxed{a^{mp}}$$

$$a^{\frac{m}{n}} \cdot a^{\frac{p}{q}}$$

$$\boxed{a^{\frac{m}{n}} \cdot a^{\frac{p}{q}}} \overset{?}{=} a^{\frac{m}{n}+\frac{p}{q}}$$

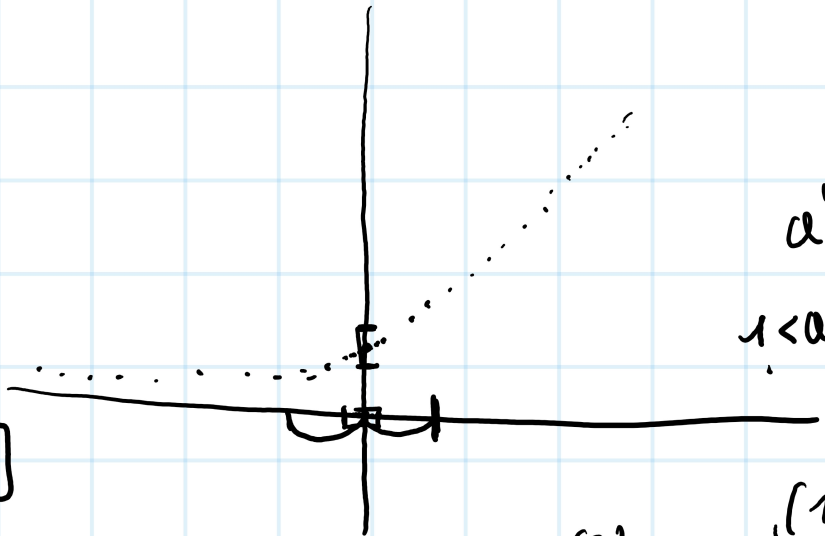
$$= \sqrt[n]{a^m} \cdot \sqrt[q]{a^p} =$$

$$= \sqrt[nq]{a^{m\cdot q}} \cdot \sqrt[nq]{a^{np}} =$$

$$= \sqrt[nq]{a^{m\cdot q} \cdot a^{n\cdot p}} =$$

$$= \sqrt[nq]{a^{m\cdot q + n\cdot p}} = a^{\frac{m\cdot q + n\cdot p}{nq}} = a^{\frac{m\cancel{q}}{n\cancel{q}} + \frac{\cancel{n}p}{\cancel{n}q}} =$$

$$= \boxed{a^{\frac{m}{n}+\frac{p}{q}}}$$



$\boxed{a>1}$

$a^0 = 1$

$1 < a^{\frac{1}{n}} < 1+\varepsilon$ $\qquad \boxed{n > \frac{a-1}{\varepsilon}}$

$\boxed{n\cdot\varepsilon > a-1}$

$\boxed{(1+\varepsilon)^n \geq 1+n\varepsilon > a}$

(?)

$\forall \varepsilon > 0 \ \exists n \in \mathbb{N}$ t.c. $\boxed{1 < a^{\frac{1}{n}} < 1+\varepsilon} \iff \boxed{a < (1+\varepsilon)^n}$

$$\boxed{(1+\varepsilon)^n \geq 1+\varepsilon\cdot n} \qquad \varepsilon > -1 \qquad n \in \mathbb{N}$$

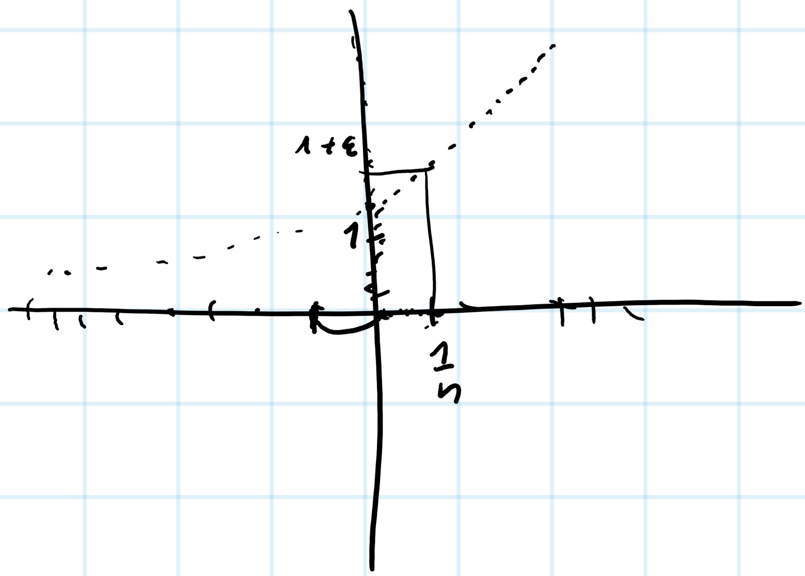
$$(1+\varepsilon)^1 \geq 1+\varepsilon\cdot 1 \quad \text{Sì}$$

$$\boxed{(1+\varepsilon)^k \geq (1+k\varepsilon)} \Rightarrow (1+\varepsilon)^{k+1} \geq 1+(k+1)\varepsilon \quad (?)$$

$$\boxed{(1+\varepsilon)^{k+1}} = (1+\varepsilon)^k\cdot(1+\varepsilon) \geq (1+k\varepsilon)\cdot(1+\varepsilon) =$$

$$= 1+k\varepsilon+\varepsilon+k\varepsilon^2 = 1+(k+1)\varepsilon+k\varepsilon^2 >$$

$$\boxed{> 1+(k+1)\varepsilon}$$



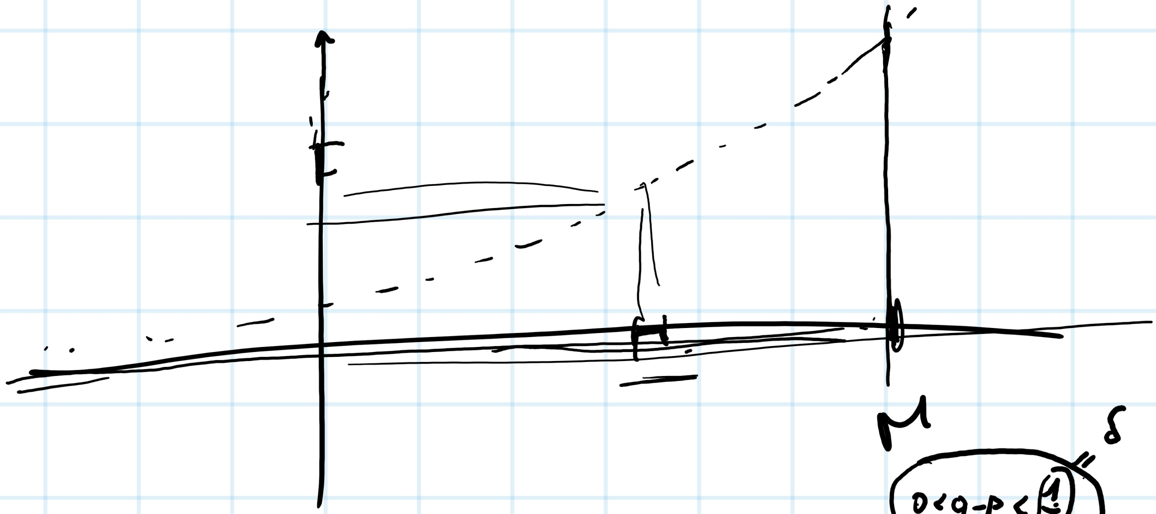
$$\boxed{\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{t.c.} \ q \in \mathbb{Q}\cap[0,\delta] \Rightarrow 1 < a^q < 1+\varepsilon}$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{t.c.} \quad q \in \mathbb{Q}\cap[-\delta,\delta] \Rightarrow |a^q - 1| < \varepsilon$$

$$\boxed{1-\varepsilon < a^q < 1+\varepsilon}$$

T. $\forall M>0$
$\forall \varepsilon>0 \ \exists \delta>0$ t.c. $\forall p,q\in\mathbb{Q}$ con $p,q\le M$ $\ |p-q|<\delta \Rightarrow |a^p-a^q|<\varepsilon$

DIM



δ

$0<q-p<\frac{1}{n}$

$\forall \varepsilon>0$

Presi p, q come sopra $(p<q)$

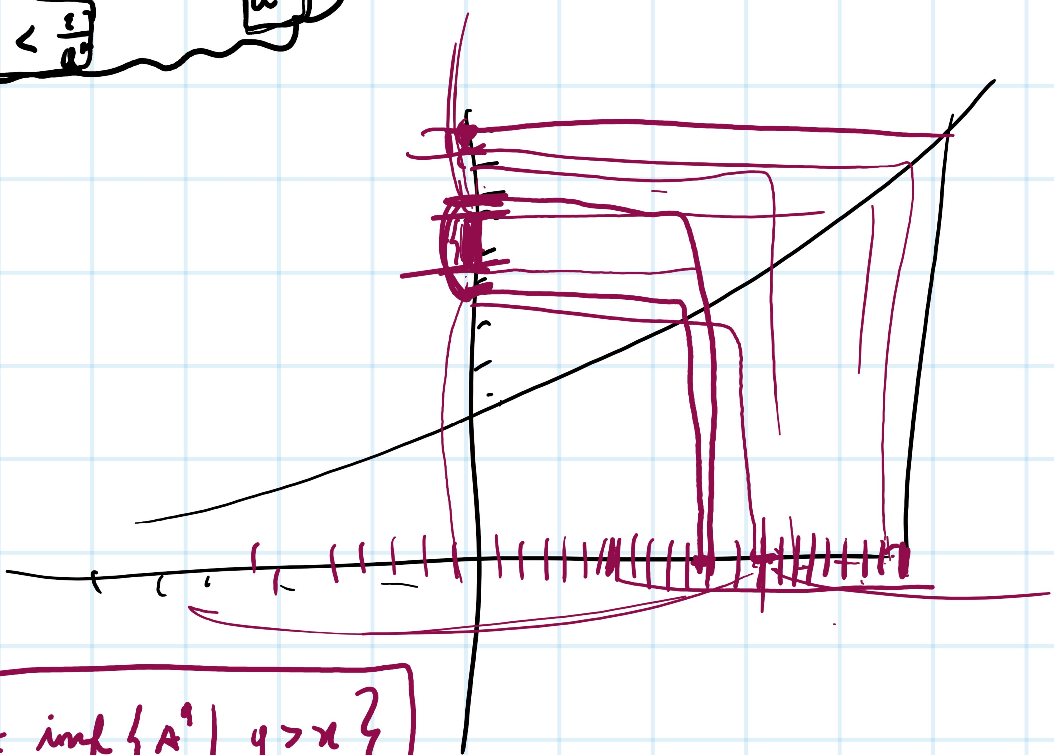
$$|a^p-a^q| = a^q-a^p = a^p\cdot(a^{q-p}-1) \le a^M(a^{q-p}-1) \le$$

$$\le \cancel{a^M}\cdot\frac{\varepsilon}{\cancel{a^M}} \le \varepsilon$$

$\forall \varepsilon>0 \ \exists n\in\mathbb{N}$ t.c.

$$1 < a^{\frac{1}{n}} < 1+\boxed{\frac{\varepsilon}{a^M}}$$

$$\boxed{0<a^{\frac{1}{n}}-1<\frac{\varepsilon}{a^M}}$$



$$\boxed{A^x = \inf\{A^q \mid q>x\}}$$

$a_n = n$

$b_n = n^2$

$c_n = 2n$

$$\frac{a_n}{b_n} = \frac{n}{n^2} = \frac{1}{n} \to 0$$

$\boxed{\text{DEF.}}$ DATE $(a_n), (b_n)$ ENTRAMBE INFINITE O INFINITESIME

DIREMO CHE

1) $a_n = o(b_n)$ SE $\lim_{n \to +\infty} \frac{a_n}{b_n} = 0$

2) $a_n \simeq b_n$ SE $\lim_{n \to +\infty} \frac{a_n}{b_n} = 1$

$a_n = \sqrt{n^2+1}$

$b_n = n$

$$\frac{a_n}{b_n} = \frac{\sqrt{n^2+1}}{n} = \sqrt{1 + \frac{1}{n^2}} \to 1$$

$\boxed{\text{T.}}$ $\boxed{a>1}$ $\boxed{\alpha>0}$ $\boxed{A>1}$ 1

$$\log_a n \ll n^\alpha \ll A^n \ll n! \ll n^n$$

$\boxed{\text{DIM}}$

1 $\frac{n^n}{n!} \to +\infty$?

$$\frac{n^n}{n!} = \frac{\overbrace{n \cdot n \cdot n \cdots n}^{n}}{\underbrace{n(n-1)(n-2)\cdots 1}_{n}} = \frac{\overbrace{n \cdot n \cdots n}^{n-1}}{\underbrace{n(n-1)\cdots 2}_{n-1}} \cdot \frac{n}{1} > n \to +\infty$$

$$\frac{A^n}{n!} \to 0 \qquad \frac{n!}{A^n} \to +\infty \quad ?$$

$$\boxed{\frac{n!}{A^n}} = \frac{n\,(n-1)(n-2)\cdots\cdots([A]+1)\cdot\overbrace{[A]\cdots\cdots 1}^{C}}{A\cdot A\cdot A\cdots\cdots A\cdot A\cdots\cdots A} =$$

$$= \frac{n}{A}\cdot\boxed{\frac{(n-1)\cdots\cdots([A]+1)}{A\cdots\cdots A}}\cdot\overbrace{\frac{[A]\cdots 1}{A\cdots A}}^{C} >$$

$$> \frac{n}{A}\cdot C = \boxed{n\cdot\frac{C}{A}} \to +\infty$$

$$n^\alpha = o(A^n) \qquad A>1 \quad \alpha>0$$

$$A = 1+\delta \quad \text{con } \delta>0$$

$$\frac{A^n}{n^\alpha} \to +\infty \quad ?$$

$$\boxed{\alpha = \frac{1}{2}}$$

$$\boxed{\frac{A^n}{\sqrt{n}}} \to +\infty \quad ?$$

$$\updownarrow$$

Bernoulli

$$\boxed{\frac{(1+\delta)^n}{\sqrt{n}}} > \frac{1+\delta\cdot n}{\sqrt{n}} = \frac{1}{\sqrt{n}} + \frac{\delta n}{\sqrt{n}} > \frac{\delta n}{\sqrt{n}} = \delta\cdot\sqrt{n} \to +\infty$$

$$\boxed{\frac{A^n}{n^\alpha}} = \frac{\left(\left(A^{\frac{1}{2\alpha}}\right)^n\right)^{2\alpha}}{\left(n^{\frac{1}{2}}\right)^{2\alpha}} = \left(\boxed{\frac{\left(A^{\frac{1}{2\alpha}}\right)^n}{\sqrt{n}}}\right)^{2\alpha} \to +\infty$$

$$\frac{B^n}{\sqrt{n}} \to +\infty$$

$\forall M > 0$

$$\left(\frac{B^n}{\sqrt{n}}\right)^{2\alpha} > \left(M^{\frac{1}{2\alpha}}\right)^{2\alpha}$$

$$\left(\frac{B^n}{\sqrt{n}}\right)^{2\alpha} > M$$

$$n^\alpha \ll A^n \qquad n \to +\infty$$

$a_n \to +\infty$

$$a_n^\alpha \ll A^{a_n}$$

$$\boxed{\frac{A^{a_n}}{a_n^\alpha}} \geq \frac{A^{\lfloor a_n \rfloor}}{(\lfloor a_n \rfloor + 1)^\alpha} =$$

$$\frac{A^k}{k^\alpha} \qquad = \frac{1}{A} \cdot \left(\frac{A^{\lfloor a_n \rfloor + 1}}{(\lfloor a_n \rfloor + 1)^\alpha}\right) \to +\infty$$

$\alpha > 0 \quad a > 1$

$$\boxed{\log_a n \ll \boxed{n^\alpha}}$$

$$\| $$

$$a^{\log_a (n^\alpha)} = a^{\alpha \cdot \log_a n} =$$

$$= \boxed{(a^\alpha)^{\log_a n}}$$

$B > 1$

$$\log_a n \qquad B^{\log_a n}$$

$$a_n \ll B^{a_n}$$