Lezione 8: Il numero di Nepero

INDICE

... DALLA LEZ. SCORSA: CONFRONTARE I SEGUENTI INFINITI:

A) $a_n = n^{10}$ $\quad b_n = 2^n$ $\quad c_n = \sqrt{3^n}$

B) $a_n = 2^{n^2}$ $\quad b_n = 16^n$ $\quad c_n = n^n$

C) $a_n = \dfrac{2^n}{n^{100}}$ $\quad b_n = \sqrt{2^n}$ $\quad c_n = 4^{\sqrt{n}}$

D) $a_n = \left(2+\frac{1}{n}\right)^n$ $\quad b_n = \left(3-\frac{1}{n}\right)^n$ $\quad c_n = \left(\log_2 n\right)^{\sqrt{n}}$

E) $a_n = \log_2(n!)$ $\quad b_n = \sqrt{1+n^2}$ $\quad c_n = \log_2(1+2^n)$

F) $a_n = (n^2)!$ $\quad b_n = (n^2)^n$ $\quad c_n = (n!)^2$

1) $\left(1+\frac{1}{n}\right)^n \to \dots$

2) GENERALIZZAZIONI

2.0) $\left(1-\frac{1}{n}\right)^n \to \dots$

2.1) $\left(1+\frac{1}{a_n}\right)^{a_n} \to \dots$

2.2) $\left(1+\frac{\lambda}{n}\right)^n \to \dots$

2.3) $\left(1+\frac{1}{a_n}\right)^{b_n} \to \dots$

3) ESEMPI

ESERCIZI

$$\log_a n \ll n^\alpha \ll A^n \ll n! \ll n^n$$

$$\boxed{a_n = 2^{n^2}} \qquad b_n = 16^n \qquad c_n = n^n$$

$$16^n = o(n^n)$$

$$16^n \ll n^n \ll 2^{n^2}$$

$$\frac{2^{n^2}}{n^n} = \left(\frac{2^n}{n}\right)^n \geq 2^n \to +\infty$$

$$\frac{2^n}{n} \to +\infty$$

$$a_n = \frac{2^n}{n^{100}} \qquad b_n = \sqrt{2^n} \qquad c_n = 4^{\sqrt{n}}$$

$$\sqrt{2^n} = (\sqrt{2})^n$$

$$\boxed{c_n \ll b_n \ll a_n}$$

$$\boxed{\frac{(\sqrt{2})^n}{4^{\sqrt{n}}}} = \left(\frac{(\sqrt{2})^{\sqrt{n}}}{4}\right)^{\sqrt{n}} > 2^{\sqrt{n}} \to +\infty$$

$$\frac{(\sqrt{2})^{\sqrt{n}}}{4} \to +\infty$$

$$(\sqrt{n})^\alpha \qquad n^{\frac{\alpha}{2}} \to +\infty$$

$$\boxed{\frac{a_n}{b_n}} = \frac{\frac{2^n}{n^{100}}}{(\sqrt{2})^n} = \frac{2^n}{n^{100}} \cdot \frac{1}{2^{\frac{n}{2}}} = \frac{2^{\frac{n}{2}}}{n^{100}} = \frac{(\sqrt{2})^n}{n^{100}} \to +\infty$$

$$a_n = (n^2)! \qquad \underbrace{b_n = (n^2)^n}_{(n^n)^2} \qquad c_n = (n!)^2$$

$$c_n \ll b_n \ll a_n$$

$$\boxed{\frac{b_n}{c_n}} = \frac{(n^n)^2}{(n!)^2} = \left(\frac{n^n}{n!}\right)^2 > \frac{n^n}{n!} \to +\infty$$

$$(n^2)! = \overbrace{n^2 \cdot (n^2-1) \cdot (n^2-2) \cdots\cdots\cdots 3 \cdot 2 \cdot 1}^{n^2}$$

$$(n^2)^n = \underbrace{n^2 \cdot n^2 \cdot n^2 \cdots n^2}_{n}$$

$(4n)!$ $\quad (n^2)!$ DEF IN n $\quad (n^2)! > (4n)!$

$$(4n)! = \overbrace{\underbrace{4n \cdot (4n-1) \cdots\cdots (2n+1)}_{2n} \cdot \underbrace{(2n) \cdot (2n-1) \cdots\cdots 3 \cdot 2 \cdot 1}_{2n}}^{4n}$$

$$(n^2)^n = n^{2n} = \underbrace{n \cdot n \cdot n \cdots\cdots n}_{2n}$$

$$\frac{(4n)!}{n^{2n}} > (2n)! \to +\infty$$

$\boxed{\text{T.}}$ DATE $a_n = \left(1+\frac{1}{n}\right)^n$ E $b_n = \left(1+\frac{1}{n}\right)^{n+1}$ ALLORA

$\exists\, e$ COMPRESO TRA 2 E 3 T.C. $a_n \to e$ CRESC.

$b_n \to e$ DECR.

$\boxed{\text{BERN.}\quad (1+x)^n \geq 1+nx \quad n\geq 1 \quad x \geq -1}$

$\boxed{\text{DIM}}$

① $\forall n \in \mathbb{N}-\{0\} \quad a_{n+1} > a_n$

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n \quad (?) \quad \text{SI}$$

$$\left(\frac{n+2}{n+1}\right)^n \cdot \left(\frac{n+2}{n+1}\right) > \left(\frac{n+1}{n}\right)^n \quad (?) \quad \text{SI}$$

$$\left(\frac{n+2}{n+1}\right)^n \cdot \left(\frac{n}{n+1}\right)^n > \frac{n+1+1-1}{n+2} \quad (?) \quad \text{SI}$$

$$\frac{n^2+2n+1-1}{n^2+2n+1}$$

$$\left(1-\frac{1}{(n+1)^2}\right)$$

$$\left(\boxed{\frac{n(n+2)}{(n+1)^2}}\right)^n > 1-\frac{1}{n+2} \quad (?) \quad \text{SI}$$

$$\left(1-\frac{1}{(n+1)^2}\right)^n > \boxed{1-\frac{1}{n+2}} \quad (?) \quad \text{SI}$$

BERN.

$$> 1+n\cdot\left(-\frac{1}{(n+1)^2}\right) =$$

$$= 1-\frac{n\cdot(n+2)}{(n+1)^2}\cdot\frac{1}{n+2} > \boxed{1-\frac{1}{n+2}}$$

(2) $\forall n \in \mathbb{N}-\{0\}$ $\quad b_n > b_{n+1}$? $\quad$ S1

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2} \quad (?)$$

$$\left(\frac{n+1}{n}\right)^{n+1} > \underbrace{\left(\frac{n+2}{n+1}\right)^{n+1}} \cdot \left(1+\frac{1}{n+1}\right) \quad (?)$$

$$\left(\frac{n+1}{n}\right)^{n+1} \cdot \underbrace{\left(\frac{n+1}{n+2}\right)^{n+1}} > \left(1+\frac{1}{n+1}\right) \quad (?)$$

$$\left(\boxed{\frac{(n+1)^2}{n(n+2)}}\right)^{n+1} > \left(1+\frac{1}{n+1}\right) \quad (?)$$

$$\boxed{\frac{n^2+2n+1}{n^2+2n}}$$

$$\boxed{1+\frac{1}{n^2+2n}}$$

$$\boxed{\left(1+\frac{1}{n^2+2n}\right)^{n+1}} > \underline{1+\frac{1}{n+1}} \quad (?)$$

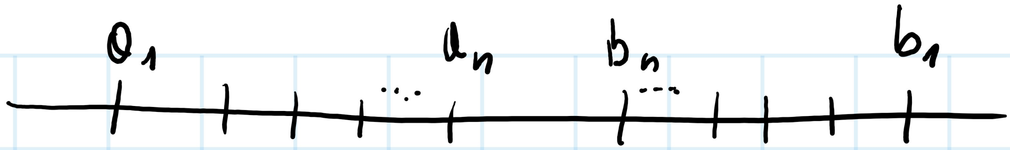
BERN. $\rightarrow$

$$> 1+(n+1)\cdot\frac{1}{n^2+2n} = 1+\overbrace{\boxed{\frac{(n+1)^2}{n^2+2n}}}^{>1}\cdot\frac{1}{n+1} >$$

$$> \boxed{1+\frac{1}{n+1}}$$

(3) $\forall n \in \mathbb{N}-\{0\}$ $\quad a_n < b_n$

$$\underline{b_n} = \left(1+\frac{1}{n}\right)^{n+1} = \overbrace{\left(1+\frac{1}{n}\right)^{n}}\cdot\left(1+\frac{1}{n}\right) = a_n \cdot \overbrace{\left(1+\frac{1}{n}\right)}^{>1} > \underline{a_n}$$



$$a_1 < a_2 < \dots < a_n < b_n < \dots < b_2 < b_1$$

(a_n) E (b_n) SONO LIMITATE PERCHÈ STANNO IN $\overset{[2,\ 4]}{[a_1, b_1]}$

QUINDI

$$a_n \to \ell \in \mathbb{R}$$

$$b_n \to L \in \mathbb{R}$$

MA $L = \ell$ PERCHÈ

$$b_n = \overbrace{\left(1+\frac{1}{n}\right)}^{\to 1} \cdot a_n \to 1 \cdot \ell = \ell$$

$$b_n \to L, \qquad a_n \to \ell$$

$$b_5 = \left(1+\frac{1}{5}\right)^6 \simeq 2{,}98\ldots$$

$$\left(1-\frac{1}{n}\right)^n \to ?$$

$$\| $$

$$\left(\frac{n-1}{n}\right)^n = \frac{1}{\left(\frac{n}{n-1}\right)^n} = \frac{1}{\left(\frac{n-1+1}{n-1}\right)^n} = \frac{1}{\underbrace{\left(1+\frac{1}{n-1}\right)^{(n-1)+1}}} \to \frac{1}{e}$$

$\boxed{k = n-1}$

$$\left(1+\frac{1}{k}\right)^{k+1} \to e$$

$$\left(1+\frac{1}{n}\right)^{n} \to e$$

$$\left(1+\frac{1}{\ln n}\right)^{\ln n}$$

$$\boxed{a_n \to +\infty} \quad \left(1+\frac{1}{a_n}\right)^{a_n} \to e$$

$$\left(1+\frac{1}{n^3+1}\right)^{n^3+1}$$

$$\underbrace{\left(1+\frac{1}{\lfloor a_n\rfloor+1}\right)^{\lfloor a_n\rfloor}}_{\to e} \leq \left(1+\frac{1}{a_n}\right)^{\lfloor a_n\rfloor} \leq \boxed{\left(1+\frac{1}{a_n}\right)^{a_n}} \leq \left(1+\frac{1}{a_n}\right)^{\lfloor a_n\rfloor+1} \leq \underbrace{\left(1+\frac{1}{\lfloor a_n\rfloor}\right)^{\lfloor a_n\rfloor+1}}_{\to e \ (S1)}$$

$$\underbrace{\left(1+\frac{1}{\lfloor a_n\rfloor+1}\right)^{\lfloor a_n\rfloor+1}}_{\to e} \cdot \underbrace{\frac{1}{1+\frac{1}{\lfloor a_n\rfloor+1}}}_{\to 1} \to e\cdot 1 = e$$

$$\left(1+\frac{7}{n}\right)^{n} = \Bigg(\underbrace{\left(1+\frac{1}{\frac{n}{7}}\right)^{\frac{n}{7}}}_{\to e}\Bigg)^{7} \to e^{7}$$

$$\boxed{\begin{array}{l} a_n \to \ell \in [0,+\infty) - \{1\} \\ b_n \to L \in \mathbb{R} \qquad a_n^{\,b_n} \to \ell^L \end{array}}$$

$$A^{a_n} \to A^{\ell} \qquad \text{SE } a_n \to \ell$$

$$f(a_n) \to f(\ell) \qquad \text{SE } a_n \to \ell$$

$$\boxed{a_n^{\,b_n}} = e^{\ln\left(a_n^{\,b_n}\right)} =$$

$$= e^{b_n \cdot \ln(a_n)} \to e^{\ln(\ell^L)} = \boxed{\ell^L}$$

$$\boxed{b_n \cdot \ln(a_n)} \to L \cdot \ln(\ell) = \boxed{\ln(\ell^L)}$$

$\lambda > 0$

$$\boxed{\left(1+\frac{\lambda}{n}\right)^n} = \left(\underbrace{\left(1+\frac{1}{\frac{n}{\lambda}}\right)^{\frac{n}{\lambda}}}_{e}\right)^{\lambda} \to \boxed{e^{\lambda}}$$

$$\left(1-\frac{1}{n}\right)^n \to \frac{1}{e}$$

$$\left(1-\frac{1}{a_n}\right)^{a_n} \to \frac{1}{e} \qquad \text{SE } a_n \to +\infty$$

$\lambda < 0$

$$\left(1+\frac{\lambda}{n}\right)^n = \left(\overbrace{\left(1-\frac{1}{\frac{n}{-\lambda}}\right)^{\frac{n}{-\lambda}}}^{\frac{1}{e}}\right)^{-\lambda} \to \left(\frac{1}{e}\right)^{-\lambda} = \boxed{e^{\lambda}}$$

$\lambda \in \mathbb{R}$ $\quad \left(1+\frac{\lambda}{n}\right)^n \to e^\lambda$

$$\overbrace{\left(1+\frac{1}{a_n}\right)^{b_n} = \left(\underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{\downarrow\, e}\right)^{\frac{b_n}{a_n}}}$$

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82 $\quad \lim_{n\to+\infty} \underbrace{\left(\frac{n+5}{n+8}\right)}\,^{\frac{n}{4}} = \left(e^{-3}\right)^{\frac{1}{4}} = \frac{1}{\sqrt[4]{e^3}}$

$$\left(\frac{n+5+3-3}{n+8}\right)^{\frac{n}{4}} = \left(1-\frac{3}{n+8}\right)^{\frac{n}{4}} =$$

$\log_a N =$

$= \frac{\log_b N}{\log_b a}$

$$= \left(\underbrace{\left(1-\frac{3}{n+8}\right)^{n+8}}_{\downarrow\, e^{-3}}\right)^{\boxed{\frac{n}{4(n+8)}} \nearrow \frac{1}{4}}$$

87 $\quad \lim_{n\to+\infty} \left(1+\boxed{\log_n 2}\right)^{\ln n} =$ $\quad \log_n 2 = \frac{\ln 2}{\ln n}$

$$= \lim_{n\to+\infty}\left(1+\frac{\ln 2}{\ln n}\right)^{\ln n} = e^{\ln 2} = 2$$

(88) $\lim_{n\to+\infty} \frac{e^n}{\left(1+\frac{1}{n^2}\right)^{n^3}}$

(89) $\lim_{n\to+\infty} \frac{e^{n^2}}{\left(1+\frac{1}{n}\right)^{n^3}}$

$\left(1+\frac{1}{n}\right)^n$

No

NO

$$\cancel{\frac{e^n}{\left(1+\frac{1}{n^2}\right)^{n^3}} = \frac{e^n}{\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^n}}$$

$$\cancel{\frac{e^{n^2}}{\left(1+\frac{1}{n}\right)^{n^3}} = \frac{e^{n^2}}{\left(\left(1+\frac{1}{n}\right)^n\right)^{n^2}}}$$

$$\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^n \le e^n \le \left(\left(1+\frac{1}{n^2}\right)^{n^2+1}\right)^n$$

$$1 = \frac{\cancel{\left(1+\frac{1}{n^2}\right)^{n^3}}}{\cancel{\left(1+\frac{1}{n^2}\right)^{n^3}}} < \boxed{\frac{e^n}{\left(1+\frac{1}{n^2}\right)^{n^3}}} < \frac{\left(1+\frac{1}{n^2}\right)^{n^3+n}}{\left(1+\frac{1}{n^2}\right)^{n^3}} = \left(1+\frac{1}{n^2}\right)^n =$$

$\downarrow$

1

$$= \left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^{\frac{1}{n}} \to e^0 = 1$$

$$\boxed{\frac{e^{n^2}}{\left(1+\frac{1}{n}\right)^{n^3}}} > \left(\frac{\left(1+\frac{1}{2n}\right)^{2n}}{\left(1+\frac{1}{n}\right)^{n}}\right)^{n^2} = \left(\frac{\left(1+\frac{1}{2n}\right)^{2}}{1+\frac{1}{n}}\right)^{n^3} =$$

$$e^{n^2} > \left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{n^2}$$

$$\frac{4n^2+4n+1}{4n^2} \cdot \frac{\cancel{n}}{n+1}$$

$$\frac{4n^2+4n+1}{4n^2+4n} =$$

$$\left(1+\frac{1}{4n^2+4n}\right)$$

$$= \left(1+\frac{1}{4n^2+4n}\right)^{n^3} =$$

$$= \left(\left(1+\frac{1}{4n^2+4n}\right)^{4n^2+4n}\right)^{\frac{n^3}{4n^2+4n}}$$

e

$+\infty$

$\rightarrow e^{+\infty} = +\infty$