Lezione 9: Calcolo di limiti

ESERCIZI SUI LIMITI (ABBUFFATA FINALE)

$\boxed{102}$ 1) $$\lim_{n\to+\infty} \frac{2^{n+\sin n}+3^{n-1}}{100^{\sqrt{n}}+\left(3+\frac{1}{n}\right)^n}$$

$\boxed{104}$ 2) $$\lim_{n\to+\infty} \frac{2^{n+1}+n^2 4^{\ln n}-2^{n-3}}{\sqrt{4^n+3}+\sqrt{3^n+4}}$$

$\boxed{106}$ 3) $$\lim_{n\to+\infty} \frac{2\cdot n^{3n}+7n\cdot(2n)!+1000^n}{10^{3n}-(2n+1)!+(n+1)^{3n}}$$

$\boxed{113}$ 4) $$\lim_{n\to+\infty} \frac{(n!)^{n+1}+((n+1)!)^n}{((n+1)!)^{n+\frac{1}{n}}+((n-1)!)^{n+2}}$$

$\boxed{107}$ 5) $$\lim_{n\to+\infty} \frac{(n^2+1)\sqrt{n}+3n^2-\sqrt{n^5+1}}{\ln^2(n+e^n)+\sqrt[n]{n!+1}}$$

$\boxed{115}$ 6) $$\lim_{n\to+\infty} \frac{(n!)^2\cdot 2^n}{(2n)!}$$

$\boxed{116}$ 7) $$\lim_{n\to+\infty} \frac{(n!)^2\cdot 3^n}{(2n)!}$$

$\boxed{117}$ 8) $$\lim_{n\to+\infty} \frac{(n!)^2\cdot 5^n}{(2n)!}$$

$\boxed{118}$ 9) $$\lim_{n\to+\infty} \frac{(n!)^2\cdot 4^n}{(2n)!}$$

$\boxed{122}$ 10) $$\lim_{n\to+\infty} \frac{(n!)^{n!}}{(n^n)!}$$

$\boxed{124}$ 11) $$\lim_{n\to+\infty} \frac{(n!)!}{n^{n^n}}$$

$\boxed{3.6 \text{ SIMU } 1}$ 12) $$\lim_{n\to+\infty} \frac{n}{\sqrt[n]{n!}}$$

FATTI: 1, 5, 10, 11, 6, 7, 8, 9

(FARE ALTRI PER CASA)

1 SOLUZIONI

$$\lim_{n\to+\infty} \frac{2^{n+\sin n} + 3^{n-1}}{100^{\sqrt{n}} + \left(3+\frac{1}{n}\right)^n} = \lim_{n\to+\infty} \frac{\frac{2^{n+\sin n}}{3^{n-1}} + 1}{\frac{100^{\sqrt{n}}}{\left(3+\frac{1}{n}\right)^n} + 1} \cdot \frac{3^{n-1}}{\left(3+\frac{1}{n}\right)^n}$$

$$\boxed{\frac{2^{n+\sin n}}{3^{n-1}}} = 3\cdot 2^{\sin n}\cdot\left(\frac{2}{3}\right)^n = \boxed{3\cdot 2^{\sin n}} \cdot \underbrace{\frac{1}{\left(\frac{3}{2}\right)^n}}_{\to 0} \to 0$$

$$0 \le \frac{100^{\sqrt{n}}}{\left(3+\frac{1}{n}\right)^n} \le \frac{100^{\sqrt{n}}}{3^n} = \Bigg(\underbrace{\boxed{\frac{100}{3^{\sqrt{n}}}}}_{\to 0}\Bigg)^{\sqrt{n}} \underset{\text{DEF. in } n}{<} \left(\frac{1}{2}\right)^{\sqrt{n}} = \underbrace{\frac{1}{2^{\sqrt{n}}}}_{\to 0}$$

$$a_n^{b_n} \to L^{\ell}$$

$$\boxed{a_n^{b_n} = e^{\ln\left(a_n^{b_n}\right)} = e^{b_n\cdot\ln(a_n)}}$$

$$\lim_{n\to+\infty} \frac{\overbrace{\frac{2^{n+\sin n}}{3^{n-1}}}^{\to 0} + 1}{\underbrace{\frac{100^{\sqrt{n}}}{\left(3+\frac{1}{n}\right)^n}}_{\to 0} + 1} \cdot \underbrace{\boxed{\frac{3^{n-1}}{\left(3+\frac{1}{n}\right)^n}}}_{\to \frac{1}{3\sqrt[3]{e}}} = \frac{0+1}{0+1}\cdot\frac{1}{3\sqrt[3]{e}} = \frac{1}{3\sqrt[3]{e}}$$

$$\frac{3^{n-1}}{\left(3+\frac{1}{n}\right)^n} = \frac{\frac{1}{3}}{\left(1+\frac{1}{3n}\right)^n} \to \frac{\frac{1}{3}}{e^{\frac{1}{3}}} = \frac{1}{3\sqrt[3]{e}}$$

$$\left(1+\frac{\lambda}{n}\right)^n \to e^{\lambda}$$

5

$$\lim_{n\to+\infty} \frac{(n^2+1)\sqrt{n} + 3n^2 - \sqrt{n^5+1}}{\ln^2(n+e^n) + \sqrt[n]{n!+1}} =$$

$$\frac{(n^2+1)\sqrt{n} - \sqrt{n^5+1}}{1} = \frac{(n^2+1)^2 \cdot n - (n^5+1)}{(n^2+1)\sqrt{n} + \sqrt{n^5+1}} =$$

$$= \frac{\cancel{n^5} + 2n^3 + n - \cancel{n^5} - 1}{\dots\dots\dots} = \boxed{\frac{2n^3+n-1}{\quad}} =$$

$$= \boxed{\frac{n^3}{n^2\sqrt{n}} \cdot \frac{2 + \frac{1}{n^2} - \frac{1}{n^3}}{1 + \frac{1}{n^2} + \sqrt{1+\frac{1}{n^5}}}}$$

$$\boxed{\frac{3n^2}{\boxed{\quad}}} = \frac{3n\sqrt{n}}{\boxed{\quad}} \to +\infty$$

$$\frac{1}{\square}$$

$$\lim_{n\to+\infty} \frac{(n^2+1)\sqrt{n} + 3n^2 - \sqrt{n^5+1}}{\ln^2(n+e^n)+\sqrt[n]{n!+1}} =$$

$$3$$

$$= \lim_{n\to+\infty} \boxed{\frac{3n^2}{\ln^2(n+e^n)}} \cdot \frac{1+\frac{(n^2+1)\sqrt{n}-\sqrt{n^5+1}}{3n^2}}{1+\frac{\sqrt[n]{n!+1}}{\ln^2(n+e^n)}} = 3\cdot\frac{1+0}{1+0} = 3$$

$$\sqrt[n]{n!+1} < \sqrt[n]{n^n} = n$$

$$(\ln(n+e^n))^2 > (\ln e^n)^2 = n^2$$

$$\frac{(\ln(n+e^n))^2}{\sqrt[n]{n!+1}} > \frac{n^2}{n} = n \to +\infty$$

$$\frac{3n^2}{(\ln(n+e^n))^2}$$

$a_n, b_n \to +\infty \quad a_n \approx b_n$

$\ln(a_n) \approx \ln(b_n)$

$e^{a_n} \not\approx e^{b_n}$

$$\frac{n+e^n}{e^n} = \frac{n}{e^n}+1 \to 0+1=1$$

$$\boxed{\ln(n+e^n)} = \ln\left(\left(\frac{n}{e^n}+1\right)\cdot e^n\right) = \ln\left(1+\frac{n}{e^n}\right)+\ln e^n =$$

$$= \left(\boxed{\ln\left(1+\frac{n}{e^n}\right)}^{\to 0} + n^{\to +\infty}\right) = \boxed{n}\cdot\boxed{\left(\frac{\boxed{\ln\left(1+\frac{n}{e^n}\right)}}{n}+1\right)}$$

$$\underbrace{\frac{\ln\left(1+\frac{n}{e^n}\right)}{n}}_{0} \qquad \underbrace{1}_{1}$$

$$\frac{3n^2}{\left(\ln\left(\underline{n+e^n}\right)\right)^2} = \frac{3\cancel{n^2}}{\left(\cancel{n}\cdot\boxed{}\right)^2} \to \frac{3}{1} = 3$$

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10 $\displaystyle\lim_{n\to+\infty} \frac{(n!)^{n!}}{(n^n)!}$

$$\boxed{\frac{n!}{n^{n-1}} \to 0}$$

$$\boxed{n! < n^{n-1}}$$
$$\boxed{< n^n}$$

$$\frac{n\cdot(n-1)\,(n-2)\ldots\, 3\cdot 2\cdot 1}{n\cdot n\cdot n\cdot\ldots\cdot n\cdot n}$$

$$\boxed{(n!)^{n!}} < (n^n)^{n^{n-1}} = n^{n^1\cdot n^{n-1}} = \boxed{n^{n^n}}$$

$$n! \ll n^n$$

$\boxed{\alpha > 1}$

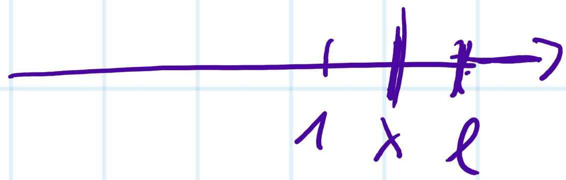
$$(n!)^\alpha \gg n^n$$

$$\frac{(n!)^{2023}}{(n^n)^{2022}} \cancel{\to\infty}$$

$$\left(\frac{(n!)^{\frac{2023}{2022}}}{n^n}\right)^{2022}$$

a_n

$$\frac{a_{n+1}}{a_n} \to \ell > 1$$



DEF. lim $\quad \frac{a_{n+1}}{a_n} > \lambda > 1$

$$\boxed{\frac{a_{n+1}}{a_n} > \lambda}$$

$$\Downarrow$$

$$a_{n+1} > \lambda a_n$$

$$a_{n_0+1} > \lambda a_{n_0}$$

$$a_{n_0+2} > \lambda a_{n_0+1} > \lambda^2 a_{n_0}$$

$$a_{n_0+3} > \lambda a_{n_0+2} > \lambda^3 a_{n_0}$$

$$\vdots$$

$\forall k \in \mathbb{N}$

$$a_{n_0+k} > \boxed{\underbrace{\lambda^k}_{\to +\infty} \underbrace{a_{n_0}}_{= c > 0}} \to +\infty$$

$\boxed{\alpha > 1}$

$$a_n = \boxed{\frac{(n!)^\alpha}{n^n}} \longrightarrow +\infty$$

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^\alpha}{(n+1)^{n+1}} \cdot \frac{n^n}{(n!)^\alpha} =$$

$$= \left(\frac{(n+1)!}{n!}\right)^\alpha \cdot \frac{n^n}{(n+1)\,\boxed{(n+1)^n}} =$$

$$= (n+1)^\alpha \cdot \frac{1}{(n+1)} \cdot \frac{1}{\left(\frac{n+1}{n}\right)^n} =$$

$$= \underbrace{(n+1)^{\alpha-1}}_{\downarrow \, +\infty} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{+\infty}{e} = +\infty$$

$$a_n \to +\infty$$

$$(n^n)! > (n^n)^{\frac{n^n}{2}} = n^{n \cdot \frac{1}{2} \cdot n^n} = (n^{n^n})^{\frac{n}{2}}$$

$$(k!)^1 \gg k^k$$

$$(k!)^2 \gg k^k$$

$$k! \gg k^{\frac{k}{2}}$$

$$\frac{(n!)^{n!}}{(n^n)!} < \frac{(n^{n^n})^1}{(n^{n^n})^{\frac{n}{2}}} = \frac{1}{(n^{n^n})^{\frac{n}{2}-1}} \to 0$$

11

$$\lim_{n \to +\infty} \frac{(n!)!}{n^{n^n}}$$

$$(n!)! \ll (n!)^{n!} \ll n^{n^n}$$

$$k! < k^k$$

6 7 8 9

$$\lim_{n\to+\infty} \boxed{\overset{a_n}{\frac{(n!)^2 A^n}{(2n)!}}} = ?$$

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2 \cdot A^{\cancel{n+1}}}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2 \cdot \cancel{A^n}} =$$

$$= \left(\frac{(n+1)!}{n!}\right)^2 \cdot \frac{(2n)!}{(2n+2)!} \cdot A =$$

$$= (n+1)^2 \cdot \frac{1}{(2n+2)(2n+1)} \cdot A =$$

$$= (n+1)^{\cancel{2}} \cdot \frac{1}{2\cancel{(n+1)}(2n+1)} \cdot A =$$

$$= \underbrace{\boxed{\frac{n+1}{n+\frac{1}{2}}} \cdot \frac{1}{4} \cdot A}_{\downarrow\ 1} \to \boxed{\frac{A}{4}}$$

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{2n+2}{2n+1} = \frac{2n+1+1}{2n+1} = \boxed{1+\frac{1}{2n+1}}$$

$$\boxed{b_n = \sqrt[4]{n}}$$

$$\frac{b_{n+1}}{b_n} = \frac{\sqrt[4]{n+1}}{\sqrt[4]{n}} = \sqrt[4]{1+\frac{1}{n}} - 1 + 1 =$$

$$= 1 + \frac{\sqrt[4]{1+\frac{1}{n}} - 1}{1} =$$

$$= 1 + \frac{\sqrt{1+\frac{1}{n}} - 1}{\sqrt[4]{1+\frac{1}{n}} + 1} =$$

$$= 1 + \frac{\cancel{1} + \frac{1}{n} \cancel{-1}}{\left(\sqrt[4]{1+\frac{1}{n}} + 1\right)\left(\sqrt{1+\frac{1}{n}} + 1\right)} =$$

$$= 1 + \frac{1}{n\underbrace{\left(\sqrt[4]{1+\frac{1}{n}} + 1\right)}\underbrace{\left(\sqrt{1+\frac{1}{n}} + 1\right)}} <$$

$$< \boxed{1 + \frac{1}{4n}}$$

$a_n =$ MIA

$b_n = \sqrt[4]{n} \to +\infty$

$$\boxed{\frac{a_{n+1}}{a_n}} = 1 + \frac{1}{2n+1} > 1 + \frac{1}{4n} > \boxed{\frac{b_{n+1}}{b_n}}$$

DA n_0 IN POI

$$\boxed{\frac{a_{n+1}}{a_n} > \frac{b_{n+1}}{b_n}}$$

$$\frac{a_{n_0+k}}{a_{n_0}} > \frac{b_{n_0+k}}{b_{n_0}} \quad (?)$$

$$\frac{a_{n_0+k}}{a_{n_0}} = \frac{a_{n_0+k}}{a_{n_0+k-1}} \cdot \frac{a_{n_0+k-1}}{a_{n_0+k-2}} \cdot \ldots \cdot \frac{a_{n_0+1}}{a_{n_0}} >$$

$$> \frac{b_{n_0+k}}{\cancel{b_{n_0+k-1}}} \cdot \frac{\cancel{b_{n_0+k-1}}}{\cancel{b_{n_0+k-2}}} \cdot \ldots \cdot \frac{\cancel{b_{n_0+1}}}{b_{n_0}} =$$

$$= \frac{b_{n_0+k}}{b_{n_0}}$$

$\forall k > 1$ $$\frac{a_{n_0+k}}{a_{n_0}} > \frac{b_{n_0+k}}{b_{n_0}}$$

$$\underbrace{\boxed{a_{n_0+k}}}_{\to +\infty} > \underbrace{\left[\underbrace{\boxed{\frac{a_{n_0}}{b_{n_0}}}}_{c>0} \cdot \underbrace{\boxed{b_{n_0+k}}}_{\to +\infty} \right]}_{\to +\infty}$$