Analisi Matematica 1 - 20 Ottobre 2023 - docente: Callegari

Lezione 11: Liminf e Limsup

INDICE

0) .... COMPLETEZZA DI $\mathbb{R}$ (FINIRE)

1) DEF. EQUIVALENTI DI LIMINF E LIMSUP

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2) ESERCIZI, TROVARE LIMINF(a_n) E LIMSUP(a_n) NEI SEGUENTI CASI:

$\boxed{2.1}$ $a_n = (-1)^n \cdot \frac{2^n+1}{2^n}$

$\boxed{2.2}$ $a_n = \left(1 + \frac{\sqrt{n^2+1} + (-1)^n \cdot n}{n^2}\right)^n$

$\boxed{2.3}$ $a_n = \sqrt{n} - \lfloor \sqrt{n} \rfloor$

$\boxed{2.4}$ $a_n = (1 + \sin n)^n$

PER CASA

(a_n) CAUCHY $\Rightarrow$ CONVERGE.

↳ LIMITATA $\rightarrow \exists (a_{n_k})$ t.c. $\underline{\underline{a_{n_k} \rightarrow \ell}} \in \mathbb{R}$

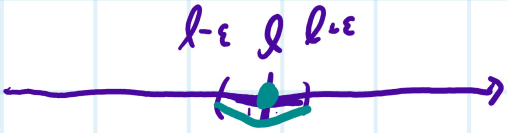
$\underline{\forall \varepsilon > 0}$ DEF inn $|a_n - \ell| < \varepsilon$

(S1)

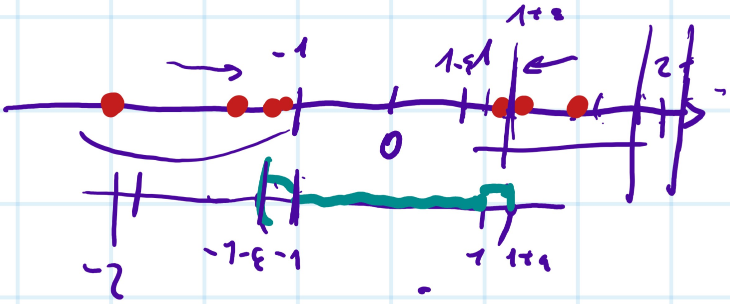
$\exists \bar{n} \in \mathbb{N}$ t.c. $n, m \underline{\geq \bar{n}} \Rightarrow |a_n - a_m| < \underline{\frac{\varepsilon}{2}}$

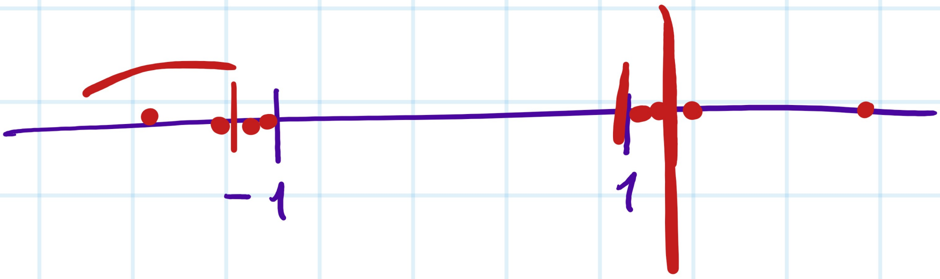
$\exists \bar{k} \in \mathbb{N}$ t.c. $\underline{k > \bar{k}}$ $\underline{|a_{n_k} - \ell| < \frac{\varepsilon}{2}}$

$n > \bar{n}$ $\underline{|a_n - \ell|} = \underline{|a_n - a_{n_k} + a_{n_k} - \ell|} \leq \underline{\overbrace{|a_n - a_{n_k}|} + \overbrace{|a_{n_k} - \ell|}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$



$a_n = (-1)^n \underbrace{\left(1+\frac{1}{n}\right)}$





LIMITATA

DEF.1 DATA (a_n) E DATI $\ell, L \in \mathbb{R}$ DIREMO CHE

$$\text{"}L = \limsup_{n\to+\infty} (a_n)\text{"} \iff \text{"}\boxed{\forall \varepsilon > 0 \quad \begin{array}{ll} \text{DEF. IN } n & a_n < L+\varepsilon \\ \text{FREQ IN } n & a_n > L-\varepsilon \end{array}}$$

$$\text{"}\ell = \liminf_{n\to+\infty} (a_n)\text{"} \iff \text{"}\forall \varepsilon > 0 \quad \begin{array}{ll} \text{DEF IN } n & a_n > \ell-\varepsilon \\ \text{FREQ IN } n & a_n < \ell+\varepsilon \end{array}$$

DEF.2 DATA (a_n) LIMITATA DIREMO CHE:

λ È MAGGIORANTE DEFINITIVO DI (a_n) SE DEF IN n $a_n \leq \lambda$

μ " MINORANTE .... $a_n \geq \mu$

A

$$L = \limsup_{n\to+\infty} (a_n) \iff L = \text{INF}\{\lambda \in \mathbb{R} \mid \lambda \text{ È MAGG. DEF. DI } (a_n)\}$$

$$\ell = \liminf_{n\to+\infty} (a_n) \iff \ell = \text{SUP}\{\mu \in \mathbb{R} \mid \mu \text{ È MIN. DEF. DI } (a_n)\}$$

OSS.

1) $A \neq \emptyset$ perché SE $-M \leq a_n \leq M$ OGNI $\lambda \geq M$ È MAGG. DEF.

2) A È INFER. LIMITATO ogni $s \leq -M$ NON È MAGG. DEF. PERCHE $\forall n \in \mathbb{N}$ $a_n > s$

QUINDI INF(A) ESISTE SEMPRE E STA IN $\mathbb{R}$ QUINDI $\limsup(a_n)$ $\exists$ SEMPRE E FINITO

(IDEM LIMINF)

L (DEF.2)

$\exists!$

ℓ IDEM

L soddisfa DEF. 2

$\Downarrow$

L soddisfa DEF. 1

$$L = \text{INF} \overbrace{\{\lambda \in \mathbb{R} \mid \lambda \text{ MAGG. DEF.}\}}^{A}$$

$\Downarrow$

POICHÈ A È UNA SEMIRETTA (ovvio...) SI HA CHE $\forall \varepsilon > 0$ $L + \varepsilon$ È MAGG. DEF. CIOÈ $\boxed{\text{DEF IN } n \quad a_n \underset{<}{\le} L + \varepsilon}$

INOLTRE SICCOME $L = \text{INF}(A)$ $\forall \varepsilon > 0$ $L - \varepsilon \notin A$ QUINDI NON È VERO CHE $\boxed{\text{DEF IN } n \quad a_n \le L - \varepsilon}$

QUINDI $\boxed{\text{FREQ IN } n \quad a_n > L - \varepsilon}$

L SODD. DEF 1

$\Downarrow$

L SODD. DEF. 2

$\boxed{L \text{ SODD. D1}} \Rightarrow$ $\forall \varepsilon > 0$ $L + \varepsilon$ È MAGG. DEF.

$L - \varepsilon$ NON È MAGG. DEF.

$\Downarrow$

INSIEME DEI MAGG. DEF. È

O $[L, +\infty)$

O $(L, +\infty)$

IN ENTRAMBI I CASI $L = \text{INF}$

$\Downarrow$

L SODD. DEF. 2

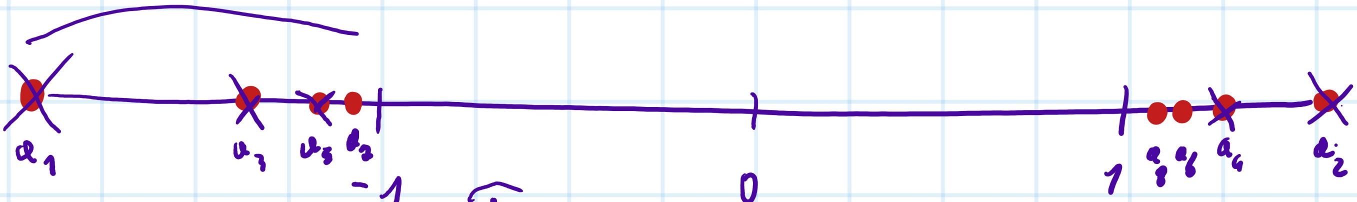
$\boxed{\text{DEF.3}}$ DATA (a_n) LIMITATA E $L, \ell \in \mathbb{R}$ DIREMO CHE

$$\text{"}L = \limsup_{n\to+\infty}(a_n)\text{"} \iff L = \lim_{n\to+\infty}\left(\sup_{k\geq n} a_k\right)$$

$$\text{"}\ell = \liminf_{n\to+\infty}(a_n)\text{"} \iff \ell = \lim_{n\to+\infty}\left(\inf_{k\geq n} a_k\right)$$

ESEMPIO

$a_n = (-1)^n\left(1+\frac{1}{n}\right)$



$$\underbrace{\lim_{n\to+\infty}\overbrace{\left(\sup_{k\geq n} a_n\right)}^{A_n}}$$

$A_1 = a_2$

$A_2 = a_2$

$A_3 = a_4$

$A_4 = a_4$

$A_5 = a_6$.

$A_6 = a_6$.

$\vdots$

$\lim A_n$

DOVE $A_n = \sup_{k\geq n} a_n$

$\boxed{n \geq m}$

$A_n = \sup_{k\geq n} \overline{a_n}$

$A_m = \sup_{k\geq m} \overline{a_n}$

$\boxed{A_n \leq A_m}$

$\overline{V} \subset \overline{W}$

$\boxed{\text{OSS.}}$ $\boxed{\begin{matrix} L_1 \\ L_2 \end{matrix}}$ quelli definiti da D1

D2

L_3 D_3

$L_3 \geq L_1$

$L_3 \leq L_1$

$\boxed{\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \text{t.c.} \quad \overset{\frown}{k \geq n_0} \quad \boxed{a_k} \leq \underline{\overline{L_1 + \varepsilon}}}$

$\boxed{A_{n_0}} = \sup_{k \geq n_0} a_k \leq L_1 + \varepsilon$

$L_3 = \lim_{n \to +\infty} \underset{\substack{\uparrow \\ \text{DECR}}}{A_n} \leq \circ \leq L_1 + \varepsilon$

$\Downarrow$

$L_3 \leq L_1$

? $\boxed{L_3 \geq L_1}$

$\rightarrow \forall \varepsilon > 0$ FREQ IN k $\underline{\underline{a_k \geq L_1 - \varepsilon}}$

$\forall n \in \mathbb{N} \quad A_n = \sup_{k \geq n} \underbrace{a_k} \geq L_1 - \varepsilon$

$\uparrow$

$\exists k > n$ t.c. $\underline{a_k \geq L_1 - \varepsilon}$

$\forall n \in \mathbb{N} \quad A_n \geq L_1 - \varepsilon$

P.A. SE $L_3 < L_1$ SAREBBE $L_3 < L_1 - \varepsilon$ PER QUALCHE ε

P.A.

$L_3 < L_1 - \varepsilon$

PRENDO I intorno di L_3 che non interseca $L_1 - \varepsilon$

$\forall \varepsilon > 0$

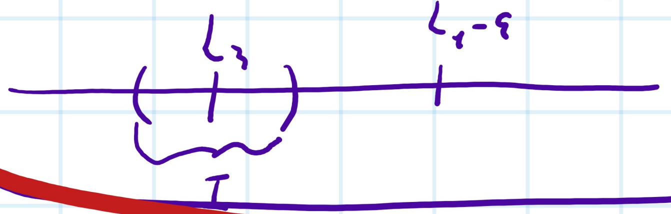
$\lim_{n \to +\infty} A_n = L_3 \geq L_1 - \varepsilon$

$L_3 \geq L_1$

$\exists$ DEF. IN N $A_n \in I$ QUINDI

DEF IN N $A_n < L_1 - \varepsilon$

ASSURDO

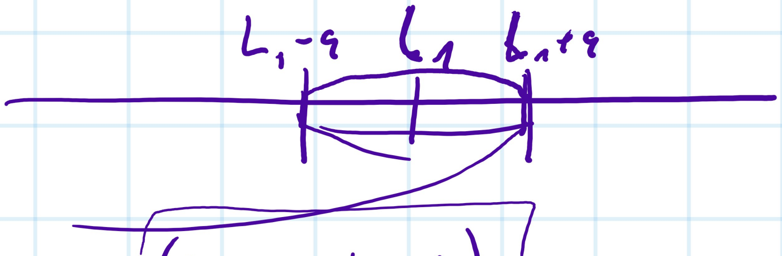


$\boxed{\text{DEF. 4}}$ DATA (a_n) LIMITATA E $\ell \in \mathbb{R}$, DIREMO CHE ℓ È UN PUNTO LIMITE DI (a_n) SE $\exists (a_{n_k})$ SSUCC. DI (a_n) T.C. $a_{n_k} \to \ell$.

CHIAMIAMO $\limsup_{n \to +\infty} (a_n)$ il punto limite più grande

$\liminf (a_n)$ · piccolo

$\forall \varepsilon > 0$



$\forall k \in \mathbb{N} \ \exists a_{n_k} \in \left(L_1 - \frac{1}{k}, L_1 + \frac{1}{k}\right)$

con $(n_{k+1} > n_k)$ $\quad |a_{n_k} - L_1| < \frac{1}{k}$