Analisi Matematica 1 - 23 Ottobre 2023 - docente: Callegari

Lezione 12: Esercizi su Liminf e Limsup

INDICE:

0) .... [DA LEZ. 11] EQUIVALENZA DI 4° DEFINIZIONE DI LIMSUP E LIMINF. CON ALTRE

1) TROVARE LIMSUP(a_n) E LIMINF(a_n) NEI SEGUENTI CASI:

1.1 $a_n = (-1)^n \frac{2^n+1}{2^n}$

1.2 $a_n = \left(1 + \frac{\sqrt{1+n^2} + (-1)^n \cdot n}{n^2}\right)^n$

1.3 $a_n = (1+\sin n)^n$

1.4 $a_n = \sqrt{n} - \lfloor \sqrt{n} \rfloor$

1.5 $(\sqrt{n} - \lfloor \sqrt{n} \rfloor)^{\sqrt{n}}$

1.6 $a_n = \sqrt{n} \cdot \sin^2 \sqrt{n}$

2) MOSTRARE CHE $\lim_{n\to+\infty} a_n = l \iff \limsup_{n\to+\infty} a_n = \liminf_{n\to+\infty} a_n = l$

3) MOSTRARE CHE $\liminf a_n + \liminf b_n \le \liminf(a_n+b_n) \le \limsup(a_n+b_n) \le \limsup a_n + \limsup b_n$

ED ESIBIRE UN ESEMPIO IN CUI VALGA LA DISUG. STRETTA.

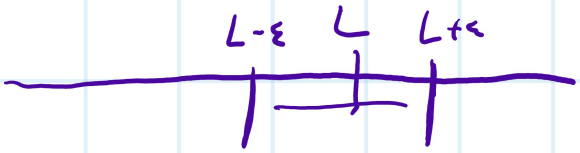
4) MOSTRARE CHE $\limsup(-a_n) = -\liminf a_n$

5) ESERCIZI DI RIEPILOGO SUI LIMITI (COMPRESO $\lim \frac{n}{\sqrt{n!}}$)

FATTI SOLO QUELLI CERCHIATI DI ROSSO

$L = \limsup(a_n) \iff \forall \varepsilon > 0$ DEF IN n $a_n < L + \varepsilon$

FREQ IN n $a_n > L - \varepsilon$



$\varepsilon = \frac{1}{k}$

FREQ IN n $\quad a_n \in \left(L - \frac{1}{k}, L + \frac{1}{k}\right)$

$\Downarrow$

n_1 t.c. $a_{n_1} \in \left(L - 1, L + 1\right)$

$n_2 > n_1$ t.c. $a_{n_2} \in \left(L - \frac{1}{2}, L + \frac{1}{2}\right)$

$\vdots$

$n_k > n_{k-1}$ t.c. $a_{n_k} \in \left(L - \frac{1}{k}, L + \frac{1}{k}\right)$

$n_{k+1} > n_k$ t.c. $a_{n_{k+1}} \in \left(L - \frac{1}{k+1}, L + \frac{1}{k+1}\right)$

$\vdots$

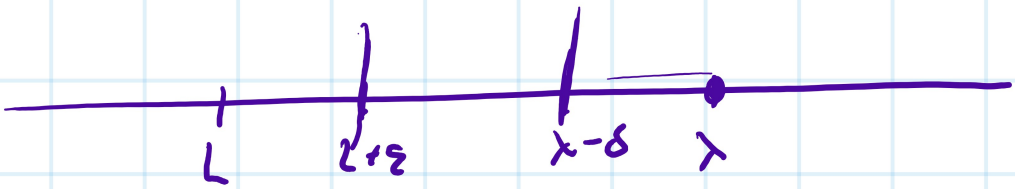
$(a_{n_k}) \quad |a_{n_k} - L| < \frac{1}{k} \to 0$

$\Rightarrow a_{n_k} \to L$

QUINDI $\forall \varepsilon > 0$ PRENDO $k_0 \in \mathbb{N}$ t.c. $\frac{1}{k_0} < \varepsilon$

e per ogni $n \geq k_0$ SI HA $|a_{n_n} - L| < \frac{1}{k} \leq \frac{1}{k_0} < \varepsilon$

L È IL PT. LIMITE PIÙ GRANDE PERCHÉ SE PRENDO $\lambda > L$,
PRENDO $\varepsilon > 0$ t.c. $L + \varepsilon < \lambda$ E $\delta > 0$ t.c. $\lambda - \delta > L + \varepsilon$



P.A.
SE CI FOSSE $a_{n_k} \to \lambda$ ALLORA $a_{n_k} > \lambda - \delta$ DEF. IMP

QUINDI $a_n > \lambda - \delta$ FREQ IN n, MA NON PUÒ
PERCHÉ $a_n < L + \varepsilon$ DEF IM n.

[... DALLA SIM.]

$$\lim_{n \to +\infty} \frac{n}{\sqrt[n]{n!}} = \dots = e \ (?)$$

(a_n) A TERMINI POSITIVI
SE $\frac{a_{n+1}}{a_n} \to l$ ALLORA $\sqrt[n]{a_n} \to l$

$$\frac{n}{\sqrt[n]{n!}} = \frac{\sqrt[n]{n^n}}{\sqrt[n]{n!}} = \sqrt[n]{\frac{n^n}{n!}} \to (?)$$

$$\lim_{n \to +\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n \to +\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n \to +\infty} \left(\frac{n+1}{n}\right)^n =$$

$$= \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

$\frac{a_{n+1}}{a_n} \to l \quad \Rightarrow \quad \sqrt[n]{a_n} \to l$ (??)

$\forall \varepsilon > 0$ DEF. IN n $\quad l-\varepsilon < \sqrt[n]{a_n} < l+\varepsilon$ (??)

$\sqrt[n]{a_n} < l+\varepsilon$??

$\forall \varepsilon > 0$ def. in n $\quad l - \frac{\varepsilon}{2} < \frac{a_{n+1}}{a_n} < l + \frac{\varepsilon}{2}$

$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N}$ t.c. $n \geq n_0 \quad \frac{a_{n+1}}{a_n} < l + \frac{\varepsilon}{2}$

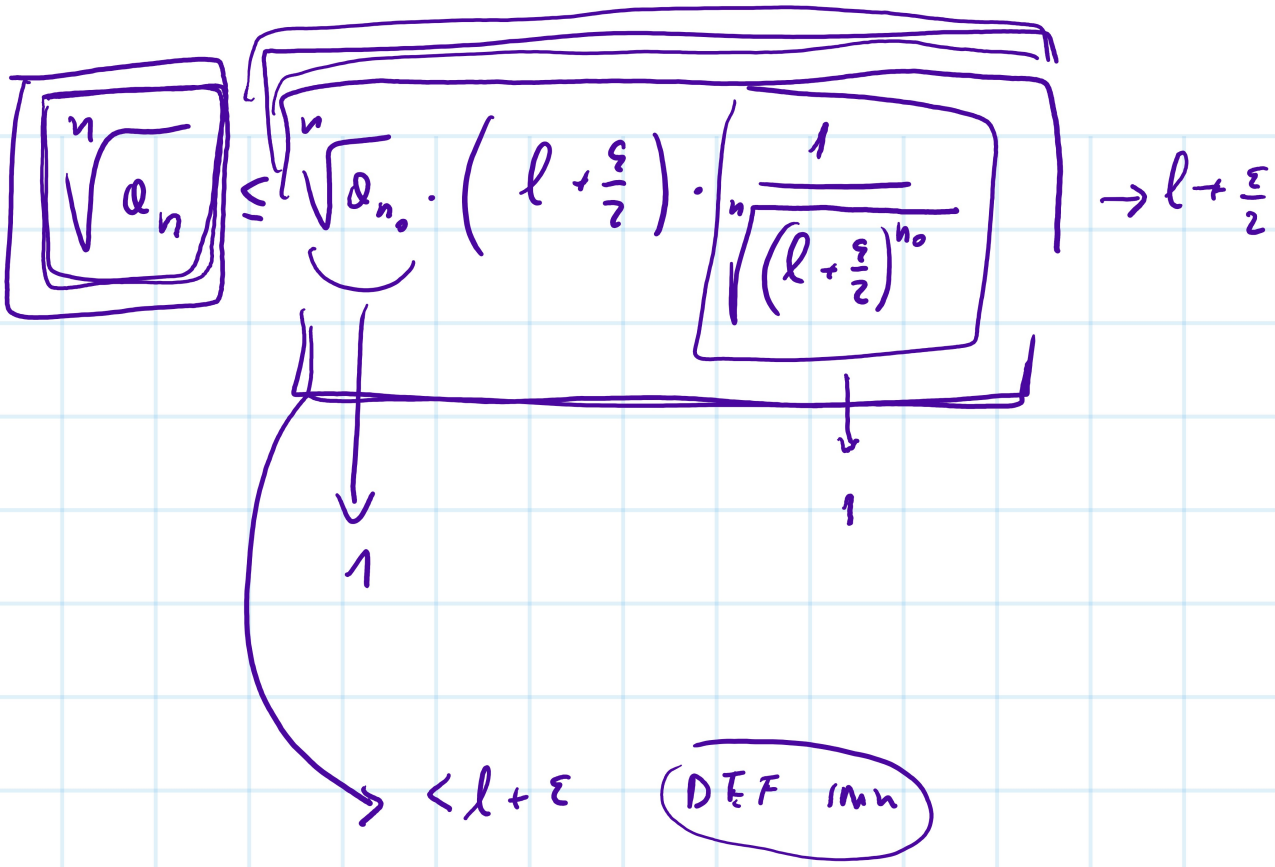
$$\sqrt[n]{a_n} = \sqrt[n]{a_{n_0} \cdot \frac{a_{n_0+1}}{a_{n_0}} \cdot \frac{a_{n_0+2}}{a_{n_0+1}} \cdot \ldots \cdot \frac{a_{n-2}}{a_{n-3}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_n}{a_{n-1}}} \leq$$

($n - n_0$ fattori)

$$\leq \sqrt[n]{a_{n_0}} \cdot \sqrt[n]{\left(l+\tfrac{\varepsilon}{2}\right) \cdot \ldots \cdot \left(l+\tfrac{\varepsilon}{2}\right)} = \sqrt[n]{a_{n_0}} \cdot \left(l+\tfrac{\varepsilon}{2}\right)^{\frac{n-n_0}{n}} =$$

($n - n_0$ fattori)

$$= \sqrt[n]{a_{n_0}} \cdot \left(l+\tfrac{\varepsilon}{2}\right)^{1-\frac{n_0}{n}}$$



$a_n = (-1)^n \frac{2^n+1}{2^n}$

$\limsup(a_n) = 1$

$\liminf(a_n) = -1$

$(-1)^n \cdot \left(1 + \frac{1}{2^n}\right)$

1ª P.T. limite PRENDO $n_n = 2k$ $\quad a_{2k} = \frac{(-1)^{2k}}{} \cdot \left(1 + \frac{1}{2^{2k}}\right) = 1 + \frac{1}{2^{2k}} \to 1$

2ª P.T. limite $\quad n_n = 2k-1 \quad a_{2k-1} \ldots \to -1$

$\forall \varepsilon > 0 \overset{\text{DEF. limn}}{\exists} (-1)^n \cdot \left(1 + \frac{1}{2^n}\right) < 1 + \varepsilon$ (?)

$\quad \hat{} \; l$

$$a_n = \left(1 + \frac{\sqrt{1+n^2} + (-1)^n n}{n^2}\right)^n$$

$n = 2k$

$$a_{2k} = \left(1 + \frac{\sqrt{1+(2k)^2} + 2k}{(2k)^2}\right)^{2k} =$$

$4k$

$$\left(1 + \frac{1}{a_k}\right)^{a_k} \to e \qquad (a_k \to +\infty)$$

$$= \left(1 + \frac{1}{\frac{(2k)^2}{\sqrt{1+(2k)^2} + 2k}}\right)$$

$$\frac{(2k)^2}{\sqrt{1+(2k)^2} + 2k} \cdot \frac{\sqrt{1+(2k)^2} + 2k}{2k}$$

$$= \left(1 + \frac{1}{a_k}\right)^{b_k} \to e$$

$$\frac{\sqrt{1+(2k)^2} + 2k}{2k} \to e^2$$

$$b_k = \frac{(2k)^2}{\sqrt{1+(2k)^2} + 2k} = \frac{4k}{\sqrt{\frac{1}{k^2} + 4 + 2}} \to +\infty$$

$\to 4$

$\to e$

$$= \sqrt{\frac{1}{(2k)^2} + 1} + 1$$

$\to 1 \quad \to 1$

$\to 2$

$$n_n = 2k-1 \qquad a_{2k-1} = \left(1 + \frac{\sqrt{1+(2k-1)^2} - (2k-1)}{(2k-1)^2}\right)^{(2k-1)} =$$

$$= \left(1 + \frac{1}{(2k-1)^2\left(\sqrt{1+(2k-1)^2} + (2k-1)\right)}\right)^{(2k-1)}$$

$$= \left(\left(1 + \frac{1}{\boxed{m}}\right)^{\boxed{m}}\right)^{\frac{1}{(2k-1)\left(\sqrt{1+(2k-1)^2}+(2k-1)\right)}}$$

$\boxed{m}$ → ∞ ; $\frac{1}{(2k-1)\left(\sqrt{1+(2k-1)^2}+(2k-1)\right)}$ → 0

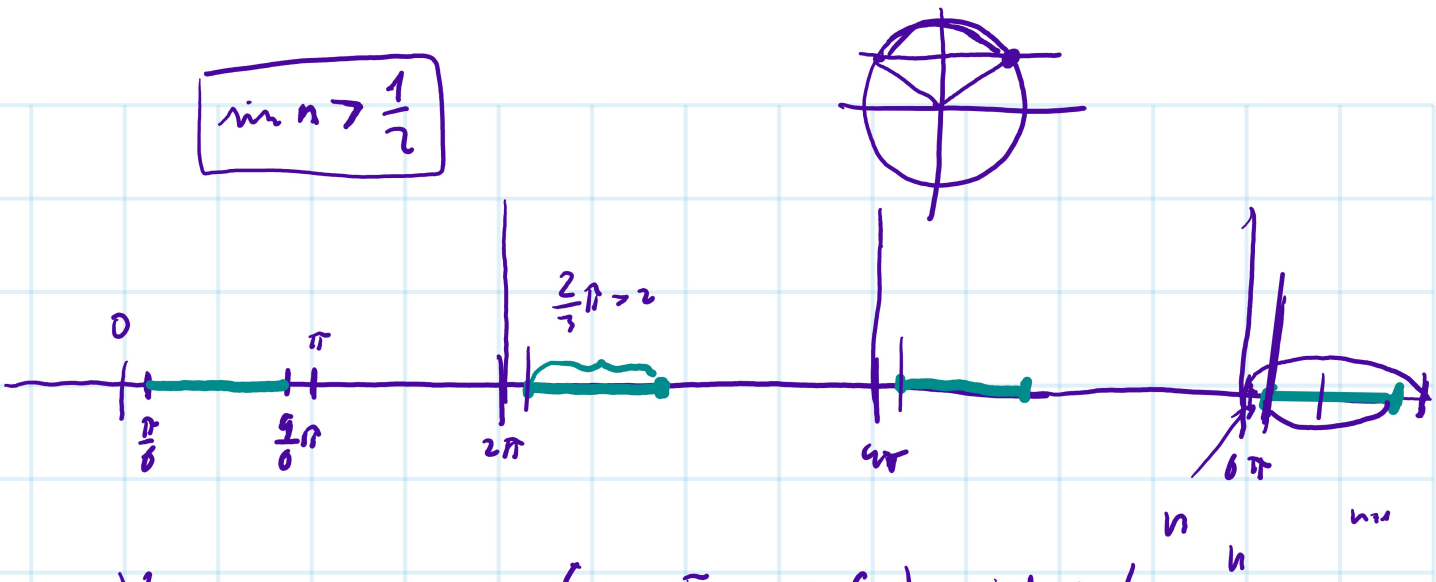
$\left(1 + \frac{1}{\boxed{m}}\right)^{\boxed{m}} \to e$; $e^0 = 1$

$a_n = (1 + \sin n)^n$

FREQ. IN n $\quad (1+\sin n)^n > \left(\frac{3}{2}\right)^n$

FREQ. IN n $\quad \alpha (1+\sin n)^n < \left(\frac{1}{2}\right)^n$

$\sin n > \frac{1}{2}$



$\forall k \in \mathbb{N} \qquad \left(2k\pi + \frac{\pi}{6}, 2k\pi + \frac{5}{6}\pi\right) \cap \mathbb{N} \neq \emptyset$

QUINDI PRENDO n_k

$\sin(n_k) > \frac{1}{2}$

$n_k > 2k\pi$

$a_n = \left(1 + \sin n\right)^n$

PRENDO $a_{n_k} = \left(1 + \sin n_k\right)^{n_k} > \left(1 + \frac{1}{2}\right)^{n_k} = \left(\frac{3}{2}\right)^{n_k} \to +\infty$

PASSO 1 MOSTRIAMO CHE $\forall k \in \mathbb{N} \quad \exists n_k \in \mathbb{N}$ t.c.

$n_k \in \left[2k\pi + \frac{\pi}{6}, 2k\pi + \frac{5}{6}\pi\right]$

DIM PRENDO $I_k = \{ n \in \mathbb{N} \mid n \leq 2k\pi + \frac{\pi}{6} \}$ È FINITO PERCHÉ

$\mathbb{N}$ NON È SUP. LIMITATO.

SIA $n_{0,k} = \max(I_k)$ ALLORA $n_{0,k} + 1 \in I_k$

PERCHÉ ALTRIMENTI SAREBBE $n_{0,k} + 1 > 2k\pi + \frac{5}{6}\pi$

$$n_{0,k} + 1 > 2k\pi + \frac{5}{6}\pi$$

$$n_{0,k} + 1 - \left(2k\pi + \frac{\pi}{6}\right) > 2k\pi + \frac{5}{6}\pi - \left(2k\pi + \frac{\pi}{6}\right)$$

$$n_{0,k} + 1 > \left(\frac{2}{3}\pi + 2k\pi + \frac{\pi}{6}\right) \text{ ma}$$

$$\boxed{n_{0,k} > 2k\pi + \left(\frac{5}{6}\pi - 1\right)} >$$

$$> 2k\pi + \frac{\pi}{6} \quad \text{(ASSURDO)}$$

QUINDI $n_{0,k+1} \in \left[2k\pi + \frac{\pi}{6}, 2k\pi + \frac{5}{6}\pi\right] \Rightarrow \boxed{\exists\, n \in \mathbb{N} \cap \left[2k\pi + \frac{\pi}{6}, 2k\pi + \frac{5}{6}\pi\right]}$

PASSO 2 PRENDO (a_{n_k}) SS. DI (a_n) AVENTE GLI INDICI DEL PASSO 1

$$a_{n_k} = \left(1 + \sin n_k\right)^{n_k} > \left(1 + \frac{1}{2}\right)^{2k\pi + \frac{\pi}{6}} > \left(1 + \frac{1}{2}\right)^{k} \to +\infty$$

$\lim (a_n) = +\infty$



$\liminf(a_n) = 0$

$\forall k \in \mathbb{N} \quad \exists\, n_k \in \mathbb{N} \cap \left[2k\pi + \frac{\pi}{6}\pi,\ 2k\pi + \frac{5}{6}\pi\right]$ $[\dots]$

$$0 < |a_{n_k}| < \left(1 + \sin n_k\right)^{n_k} < \left(\frac{1}{2}\right)^{n_k} < \left(\frac{1}{2}\right)^{2k\pi + \frac{\pi}{6}\pi} < \left(\frac{1}{2}\right)^{k} \to 0$$

$\Downarrow$

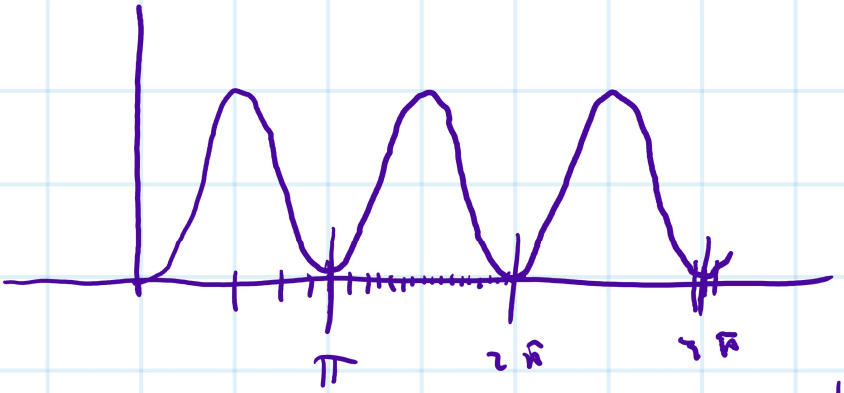
$\liminf = 0$

$a_n = \sqrt{n} \cdot \sin^2 \sqrt{n}$ $\qquad \sqrt{4}$ $\qquad \limsup(a_n) = +\infty$ $\qquad \liminf(a_n) = 0$

(n_k)

$f(x) = x \sin^2 x$

$f(\sqrt{n})$



n_3

$\forall k \in \mathbb{N}$ PRENDO n_k t.c.

$\sqrt{n_k} < k\pi < \sqrt{n_k + 1}$

$$\left(\sin\sqrt{n_k}\right)^2 =$$

$$= \left(\sin\sqrt{n_k} - \sin(k\pi)^2\right)^2 \leq$$

$$\leq \left(\sin\sqrt{n_k} - \sin\sqrt{n_k+1}\right)^2 \leq$$

$$\leq \left(\sqrt{n_k+1} - \sqrt{n_k}\right)^2 = \frac{1}{\left(\sqrt{n_k+1} + \sqrt{n_k}\right)^2}$$

$$0 \leq \sqrt{n_k}\left(\sin\sqrt{n_k}\right)^2 \leq$$