Lezione 14: Limiti di funzioni da R in R (II parte)

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A $\lim_{x\to+\infty} \sin x$

B $\lim_{x\to 0} \frac{1}{x}$

C $\lim_{x\to 0^+} \frac{1}{x}$

D $\lim_{x\to+\infty} \frac{\sin x}{x}$

E $\lim_{x\to+\infty} \frac{\arctan(\tan x)}{x}$

F $\lim_{x\to+\infty} \left(1+\frac{\sin x}{x}\right)^x$

G $\lim_{x\to 0} \frac{\sin 3x}{\sqrt{1+x}-1}$

H $\lim_{x\to 0} \frac{\cos(\sin x)-1}{(e^{3x}-1)\ln(1+4x)}$

I $\lim_{x\to 0} \frac{\sqrt{1+x^2}-\cos x}{e^{x^2}-e^x}$

J $\lim_{x\to 0} \frac{\tan(\tan x)-\sin(\sin x)}{x^3}$

K $\lim_{x\to 0} \frac{\tan(\tan(\tan x))-\sin(\sin(\sin x))}{x^3}$

L GENERALIZZARE J E K

T (Perm. Segno)

$(*)$ (<0)

DATI $A \subset \mathbb{R}$, $x_0 \in \mathbb{R}^*$ DI ACC. PER A, $f: A \to \mathbb{R}$ t.c. $\boxed{\lim_{x \to x_0} f(x) = \ell > 0}$

ALLORA $\exists I$ INTORNO DI x_0 t.c. $\forall x \in I \cap (A - \{x_0\})$ SI HA $f(x) > 0$

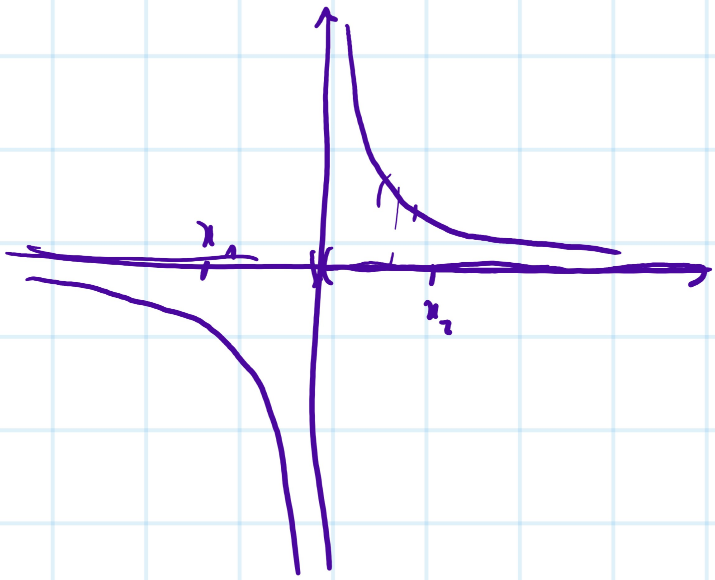
$\boxed{\text{DIM}}$ PRENDO J INTORNO DI ℓ t.c. $0 \notin J$ QUINDI $\forall y \in J$ SI HA $y > 0$.

DA $(*)$ SEGUE CHE $\exists I$ INTORNO DI x_0 t.c. $\forall x \in I \cap (A - \{x_0\})$ SI HA $f(x) \in J$, E QUINDI $f(x) > 0$.

$\boxed{\text{DEF.}}$ DATI $A \subset \mathbb{R}$, $f: A \to \mathbb{R}$ DIREMO CHE f È CRESCENTE SU A (STR. CRESC.)

SE $\forall x_1, x_2 \in A$ SE $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ $(<)$

$f(x) = \frac{1}{x}$



T. (LIM. DI F. CRESCENTI)

DATI $A \subset \mathbb{R}$, $f: A \to \mathbb{R}$ CRESCENTE, SIA $b = \sup A < +\infty$, SUPPONIAMO INOLTRE CHE b SIA DI ACC. PER A.

ALLORA $\displaystyle\lim_{x \to b^-} f(x) = \boxed{\sup_{x \in A - \{b\}} f(x)} = \lambda$.

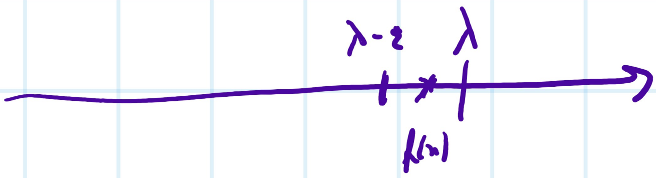
$\boxed{\text{DIM}}$ DISTINGUIAMO 2 CASI: $\lambda = +\infty$ $\quad \lambda \in \mathbb{R}$

$\boxed{\text{I° CASO}}$ $\lambda \in \mathbb{R}$

$$\lambda \geq f(x) \qquad \forall x \in A - \{b\}$$

MA

$$\forall \varepsilon > 0 \quad \exists x \in A - \{b\} \text{ t.c. } \lambda - \varepsilon < f(x)$$



$$\forall \varepsilon > 0 \quad \exists x_0 \in A - \{b\} \text{ t.c. } \quad \lambda - \varepsilon < f(x_0) \leq \lambda$$

MA ALLORA $\forall x \in A - \{b\}$ CON $x > x_0$ SI HA $f(x) \geq f(x_0) > \lambda - \varepsilon$

QUINDI $\boxed{\text{PRESO } \delta = b - x_0}$

$\forall \varepsilon > 0$ $\quad x \in (A - \{b\}) \cap I_b(\delta) \Rightarrow |f(x) - \lambda| < \varepsilon$

$\boxed{\text{II° CASO}}$ $\lambda = +\infty$

$$\forall M > 0 \quad \exists x_0 \in A - \{b\} \text{ T.C. } f(x_0) > M$$

$$y = e^x$$

$$\boxed{\lim_{x\to-\infty} \frac{\sin(e^x)}{e^x}} = \lim_{y\to 0} \frac{\sin y}{y} = 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

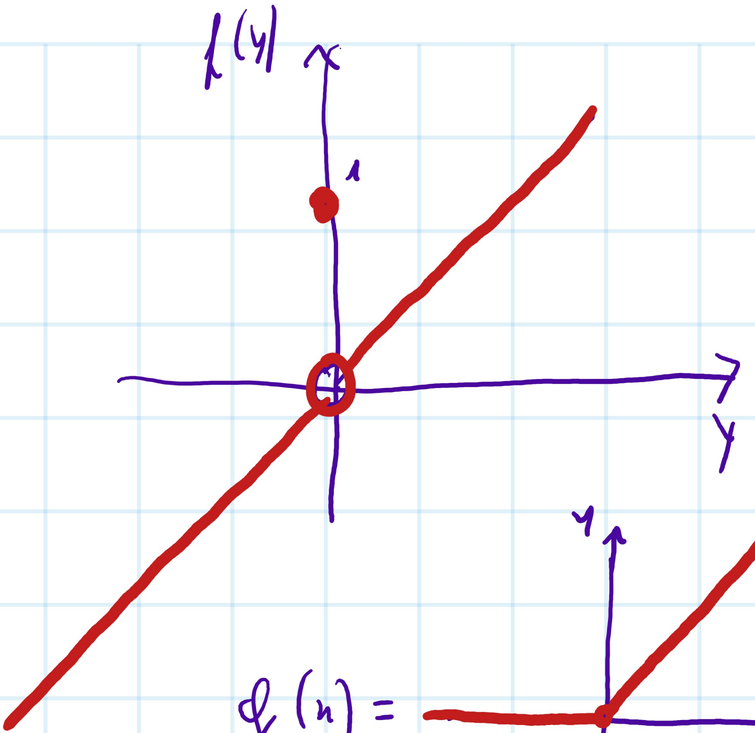
$$f(y) = \frac{\sin y}{y} \qquad g(x) = e^x$$

$$\boxed{\lim_{x\to x_0} f(\underbrace{g(x)}_{y \to y_0})} = \lim_{y\to y_0} f(y) =$$

$$\lim_{x\to x_0} g(x) = y_0$$

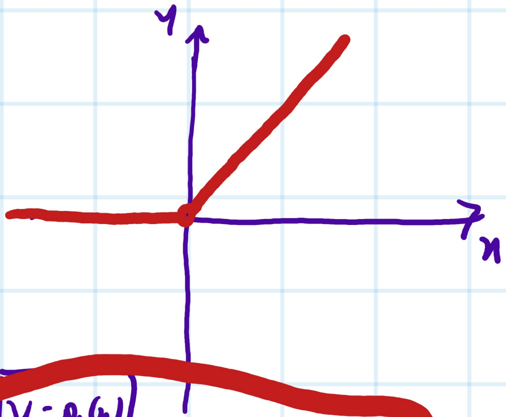
ES. CATTIVO

$$f(y) = \begin{cases} y & \text{se } y \neq 0 \\ 1 & \text{se } y = 0 \end{cases} \qquad g(x) = \frac{x+|x|}{2}$$



$\lim_{y\to 0} f(y) = 0$

$\lim_{x\to 0} g(x) = 0$

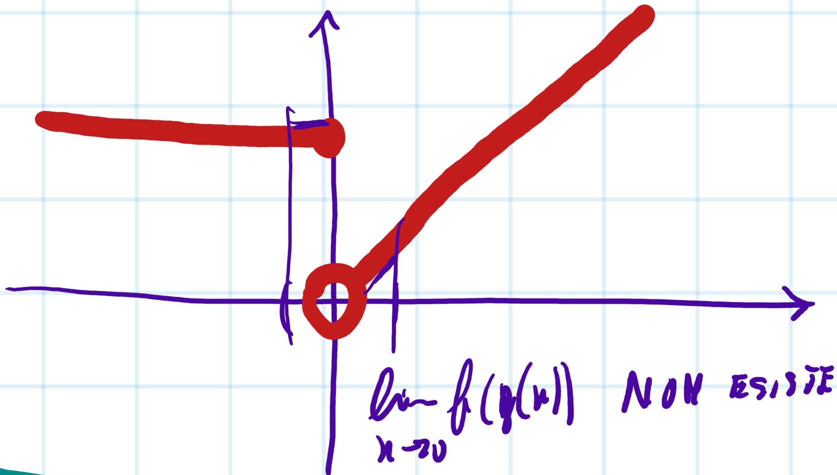
$g(x) =$ 

NO

$\lim_{x\to 0} f(g(x)) \overset{y=g(x)}{=} \lim_{y\to 0} f(y) = 0$

SI

$$f(g(x)) = \begin{cases} x & x > 0 \\ 1 & x \le 0 \end{cases}$$



$\lim_{x\to 0} f(g(x))$ NON ESISTE

T. (LIM. DI f. COMPOSTA)

DATI $A, B \subset \mathbb{R}$, x_0 DI ACC. PER A, y_0 DI ACC PER B, $g: A \to B$

$f: B \to \mathbb{R}$, T.C. $\lim_{x \to x_0} g(x) = y_0$, $\underline{\lim_{y \to y_0} f(y) = l}$

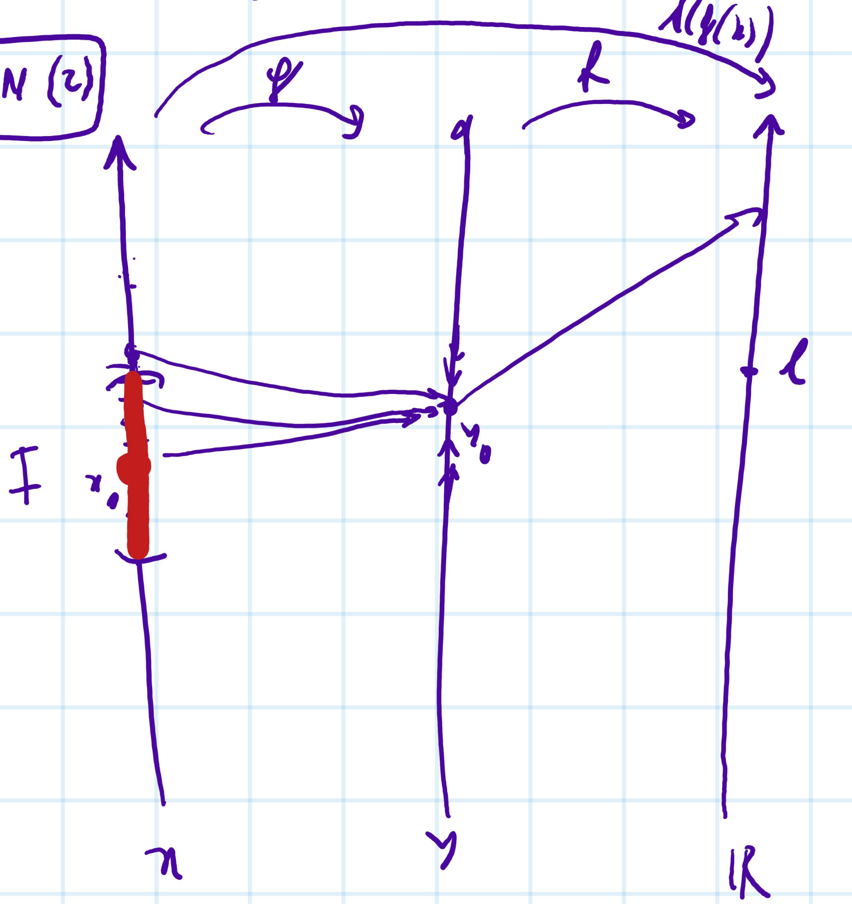
INOLTRE VALGA UNA DELLE SEGUENTI:

1) $f(y_0) = l$

2) $\exists I$ INTORNO DI x_0 T.C. $\forall x \in I - \{x_0\}$ $g(x) \neq y_0$

ALLORA $\lim_{x \to x_0} f(g(x)) = \lim_{y \to y_0} f(y) = l$

$\boxed{\text{DIM}}$ $\boxed{\text{CON (2)}}$



(CON $a_n \to x_0$) SI HA

$\underbrace{\forall (a_n) \text{ A VALORI IN } A - \{x_0\}}_{\Uparrow}$ $f(g(a_n)) \to l$ (?)

$g(a_n) \to y_0$

$\big(g(a_n)\big)$ È A VALORI $B - \{y_0\}$

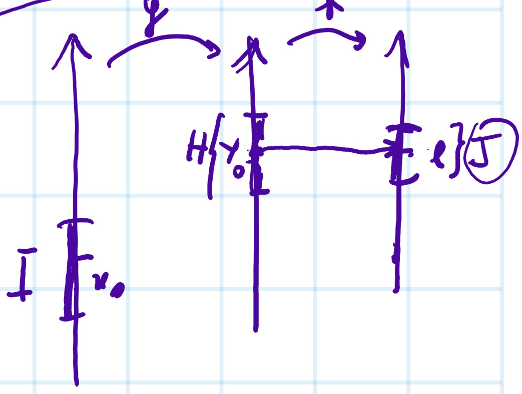
$f(g(a_n)) \to l$

$\boxed{\text{CON } (1)}$

(?) $\boxed{\forall J \text{ INTORNO DI } l}$ $\exists I$ INTORNO DI x_0 t.c. $\forall x \in I \cap (A - \{x_0\})$ SI HA $f(g(x)) \in J$

$\exists H$ INTORNO DI y_0 T.C. $\forall y \in H \cap (B - \cancel{\{y_0\}})$ SI HA $\boxed{f(y) \subset J}$

$\boxed{f(y_0) = l}$



POICHÉ $\lim_{x \to x_0} g(x) = y_0$, $\exists I$ INTORNO di x_0

T.C. $x \in I \cap (A - \{x_0\}) \Rightarrow g(x) \in H$

$\Downarrow$

$f(g(x)) \in J$

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x\to 0} \frac{\tan x}{x} = 1$

3. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

4. $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}$

5. $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$

6. $\lim_{x\to 0} \frac{\arctan x}{x} = 1$

7. $\lim_{x\to \pm\infty} \left(1+\frac{1}{x}\right)^x = e$

8. $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

9. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

10. $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

11. $\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$

7 $\boxed{\lim_{n\to \pm\infty} \boxed{\left(1+\frac{1}{a_n}\right)^{a_n}} = e}$

SE $a_n \to +\infty$

$f(a_n)$ CON $f(x) = \left(1+\frac{1}{x}\right)^x$

$\boxed{\lim_{x\to +\infty} f(x) = e}$

$\Updownarrow$

$\boxed{\lim_{n\to +\infty} f(a_n) = e \quad \forall a_n \to +\infty}$

$$\lim_{x\to +\infty} \left(1-\frac{1}{x}\right)^x = \frac{1}{e} \quad (??)$$

$$\| $$

$$\lim_{x\to +\infty} \frac{1}{\left(\frac{x}{x-1}\right)^x} = \lim_{x\to +\infty} \frac{1}{\underbrace{\left(1+\frac{1}{x-1}\right)^{x-1}}_{\downarrow\, e} \cdot \underbrace{\left(1+\frac{1}{x-1}\right)}_{\downarrow\, 1}} = \boxed{\frac{1}{e}}$$

$\frac{x-1+1}{x-1}$

$y = x - 1$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x-1}\right)^{x-1} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y} = e$$

$f(g(x))$

$f(y) = \left(1+\frac{1}{y}\right)^{y} \to e$

$g(x) = x - 1 \quad \to +\infty$

SENZA TOCCARLO

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to-\infty}\left(1-\frac{1}{(-x)}\right)^{x} =$$

$$= \lim_{x\to-\infty}\frac{1}{\left(1-\frac{1}{(-x)}\right)^{(-x)}} = \frac{1}{\frac{1}{e}} = e$$

$f(g(x))$

$y\to+\infty \quad f(y) = \left(1-\frac{1}{y}\right)^{y} \to \frac{1}{e}$

$x\to-\infty \quad g(x) = -x \to +\infty$

SENZA TOCCARLO

$$\lim_{x\to-\infty} f(g(x)) = \lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{y} = \frac{1}{e}$$

$y = g(x)$

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = \lim_{x\to 0^+}\left(1+\frac{1}{\frac{1}{x}}\right)^{\left(\frac{1}{x}\right)} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y} = e$$

0^- $\quad$ 0^- $\quad$ $(-\infty)$

$y = \frac{1}{x}$

$f(g(x))$

$g(x) = \frac{1}{x}$ $(\to -\infty)$

$f(y) = \left(1+\frac{1}{y}\right)^{y}$