Lezione 15: Limiti notevoli - Esercizi

INDICE

1) LIMITI NOTEVOLI

2) ESERCIZI. CALCOLARE O MOSTRARE CHE NON ESISTONO I SEGUENTI LIMITI:

$$-1 < \sin x < 1$$

$$-\frac{1}{x} < \frac{\sin x}{x} < +\frac{1}{x}$$

A) $\lim\limits_{x\to+\infty} \sin x$

B) $\lim\limits_{x\to 0} \frac{1}{x}$

C) $\lim\limits_{x\to 0^+} \frac{1}{x}$

D) $\lim\limits_{x\to+\infty} \frac{\sin x}{x}$

E) $\lim\limits_{x\to+\infty} \frac{\arctan(\tan x)}{x}$

E.2) $\lim\limits_{x\to+\infty} \frac{\tan(\arctan x)}{x}$

F) $\lim\limits_{x\to+\infty} \left(1+\frac{\sin x}{x}\right)^x$

G) $\lim\limits_{x\to 0} \frac{\sin 3x}{\sqrt{1+x}-1}$

H) $\lim\limits_{x\to 0} \frac{\cos(\sin x)-1}{(e^{3x}-1)\ln(1+4x)}$

I) $\lim\limits_{x\to 0} \frac{\sqrt{1+x^2}-\cos x}{e^{x^2}-e^x}$

J) $\lim\limits_{x\to 1} \frac{4^x-2^{x+1}}{\ln x}$

K) $\lim\limits_{x\to+\infty} |\sin(\sin x)|^x$

L) $\lim\limits_{x\to+\infty} |\cos(\cos x)|^x$

M) $\lim\limits_{x\to -\frac{\pi}{2}} \frac{\sin x+1}{\cos 4x-1}$

N) $\lim\limits_{x\to+\infty} \left(\cos\frac{1}{x}\right)^{x^2}$

O) $\lim\limits_{x\to 0^+} \left(\frac{\pi}{2}+\arctan(\ln x)\right)\cdot \ln\frac{1}{x}$

P) $\lim\limits_{x\to 0^-} \frac{\tan x - \sin(x+x^2)}{e^{x^2}-\cos x+e^{\frac{1}{x}}}$

Q) $\lim\limits_{x\to+\infty} \left|\sin x \cdot \cos\sqrt{x^2+1}\right|^x$

R) $\lim\limits_{x\to 0} \frac{\tan(\tan x)-\sin(\sin x)}{x^3}$

S) $\lim\limits_{x\to+\infty} \frac{\tan(\tan(\tan x))-\sin(\sin(\sin x))}{x^3}$

T) $$\lim_{x\to 0} \frac{\overbrace{\tan(\tan(\cdots(\tan x)\cdots))}^{2024} - \overbrace{\sin(\sin(\cdots(\sin x)\cdots))}^{2024}}{x^3}$$

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x\to 0} \frac{\tan x}{x} = 1$

3. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

4. $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}$

5. $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$

6. $\lim_{x\to 0} \frac{\arctan x}{x} = 1$

7. $\lim_{x\to \pm\infty} \left(1+\frac{1}{x}\right)^x = e$

8. $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

9. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

10. $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

11. $\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$

$\boxed{\text{7 E 8 GIÀ FATTI LEZ. SCORSA}}$

9 $$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \ln\Big(\underbrace{(1+x)^{\frac{1}{x}}}_{\downarrow\, e}\Big) = \ln e = 1$$

10 $$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \boxed{\frac{e^x-1}{\ln(1+(e^x-1))}} = \lim_{y\to 0} \frac{y}{\ln(1+y)} =$$

$\ln(e^x+1-1)$

$y = e^x - 1$

$f(y) = \frac{y}{\ln(1+y)}$

$g(x) = e^x - 1$

$f(g(x))$

$$= \lim_{y\to 0} \frac{1}{\frac{\ln(1+y)}{y}} = 1$$

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$$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\lim_{x\to 0}\frac{e^{\ln((1+x)^\alpha)}-1}{x}=$$

$$=\lim_{x\to 0}\boxed{\frac{e^{\alpha\ln(1+x)}-1}{\alpha\ln(1+x)}}\cdot\alpha\,\boxed{\frac{\ln(1+x)}{x}}=1\cdot\alpha\cdot 1=\alpha$$

$$\frac{\ln(1+x)}{x}\to 1$$

$$\boxed{f(y)=\frac{e^y-1}{y}}\qquad g(x)=\alpha\ln(1+x)$$

$$\boxed{f(g(x))}$$

$$\lim_{x\to 0}f(g(x))=\lim_{y\to 0}f(y)=1$$

1

$$\lim_{x\to 0^+}\frac{\sin x}{x}=1$$

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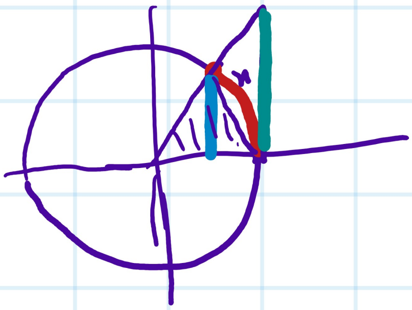
$$\boxed{0<x<\frac{\pi}{2}}$$

$$\boxed{\sin x<x<\tan x}$$



$$1<\frac{x}{\sin x}<\frac{1}{\cos x}$$

$$1\to 1,\qquad \frac{x}{\sin x}\to 1,\qquad \frac{1}{\cos x}\to 1$$

2

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\boxed{\frac{1}{\cos x}}=1$$

$$\frac{\sin x}{x}\to 1,\qquad \frac{1}{\cos x}\to 1$$

③ $$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\boxed{(1-\cos x)(1+\cos x)}}{x^2}\cdot\frac{1}{1+\cos x}=$$

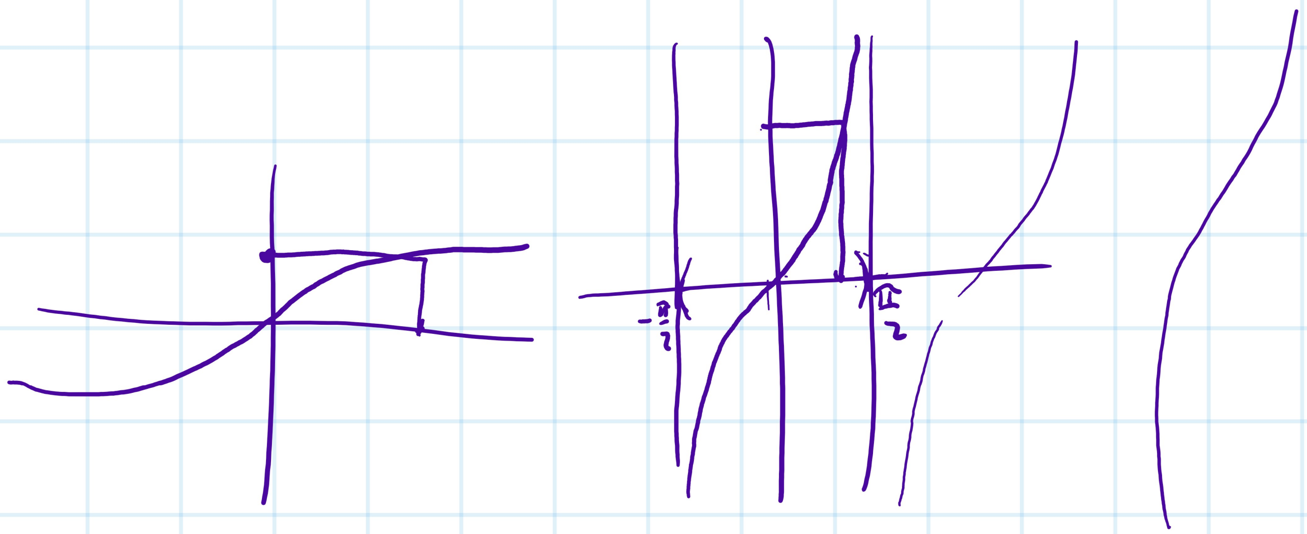
$$=\lim_{x\to 0}\boxed{\frac{\sin^2 x}{x^2}}\cdot\frac{1}{1+\cos x}=(1)^2\cdot\frac{1}{2}=\frac{1}{2}$$

$$\left(\frac{\sin x}{x}\right)^2$$

④ $$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x\cdot\left(\frac{1}{\cos x}-1\right)}{x^3}=\lim_{x\to 0}\underbrace{\frac{\sin x}{x}}_{1}\cdot\underbrace{\frac{1-\cos x}{x^2}}_{\frac{1}{2}}\cdot\underbrace{\frac{1}{\cos x}}_{1}=\frac{1}{2}$$

⑥ $$\lim_{x\to 0}\frac{\arctan x}{x}=\lim_{x\to 0}\frac{\arctan x}{\tan(\arctan x)}=\lim_{y\to 0}\frac{y}{\tan y}=1$$

$y=\arctan x$



(1) $\lim_{x\to+\infty} \sin x$ N.E.

$a_n = \frac{\pi}{2} + 2n\pi \to +\infty \quad f(a_n) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$

$b_n = \frac{3}{2}\pi + 2n\pi \to +\infty \quad f(b_n) = \sin\left(\frac{3}{2}\pi + 2n\pi\right) = -1$

$f(a_n) \to 1$

$f(b_n) \to -1$

(2) $\lim_{x\to 0} \frac{1}{x} = $ N.E

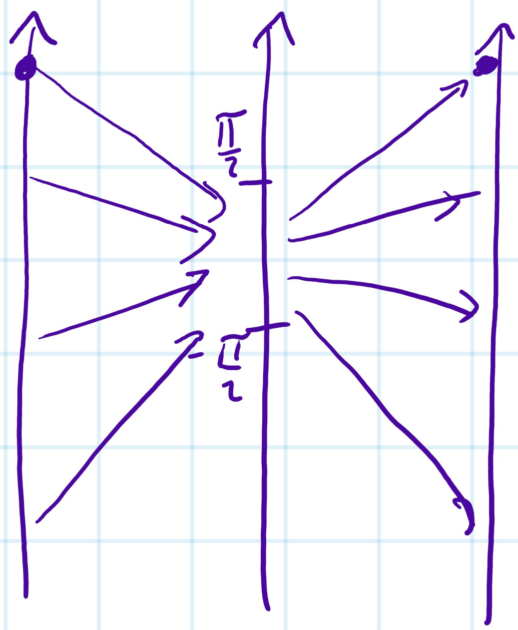
$a_n = \frac{1}{n} \quad a_n \to 0 \quad f(a_n) = \frac{1}{\frac{1}{n}} = n \to +\infty$

$b_n = -\frac{1}{n} \quad b_n \to 0 \quad f(b_n) = \frac{1}{-\frac{1}{n}} = -n \to -\infty$

(3) $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

$\lim_{x\to+\infty} \frac{\arctan(\tan x)}{x} =$

$\lim_{x\to+\infty} \frac{\tan(\arctan x)}{x} = \lim_{x\to+\infty} \frac{x}{x} = 1$



$\boxed{\forall x \in \mathbb{R} \quad \tan(\arctan x) = x}$

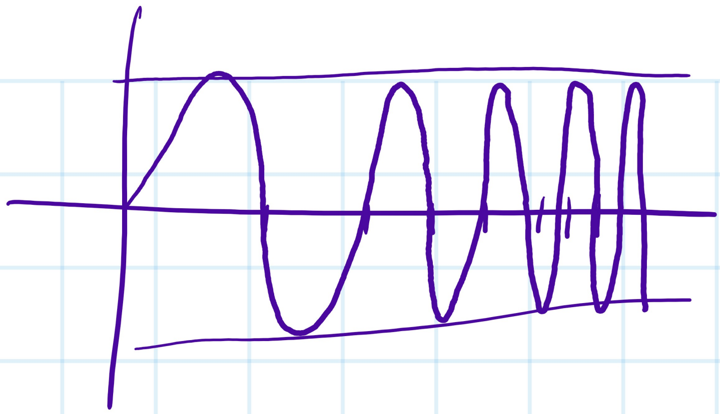
$\forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \boxed{\arctan(\tan x)} = x$

$x^2 = k\pi$

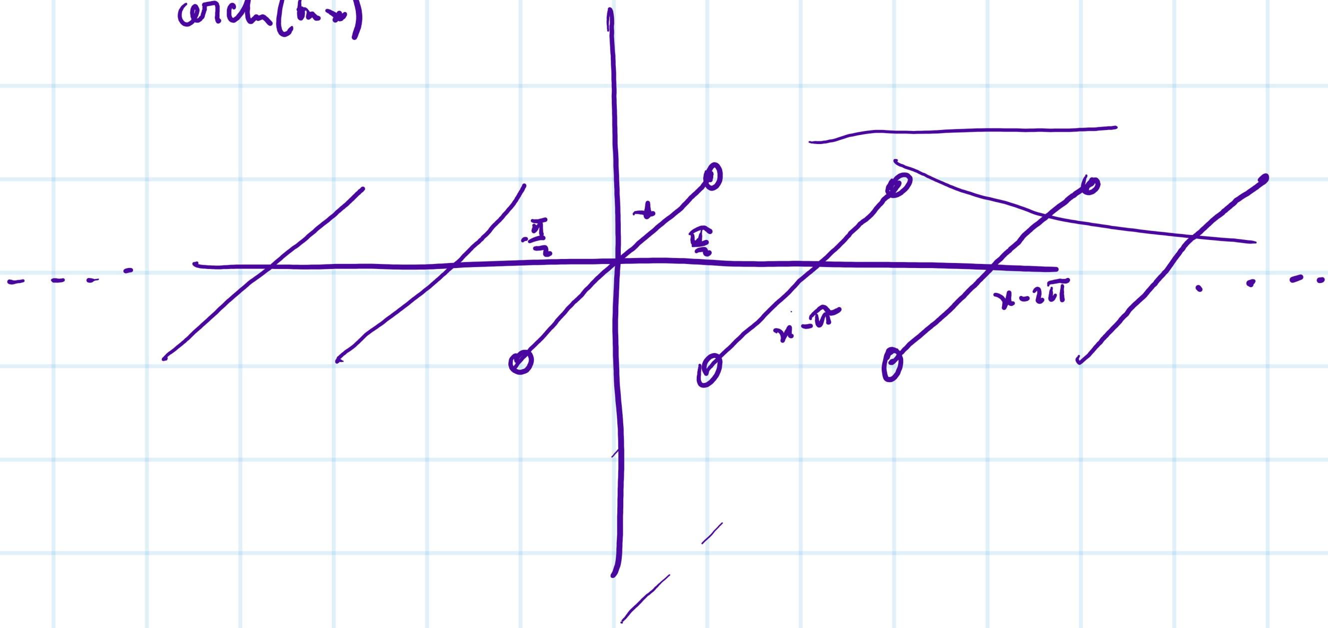
$x = \sqrt{k\pi}$

$\sin(x^2)$

$(\sin x)^2$



$\arctan(\tan x)$



$$\frac{-\frac{\pi}{2}}{x} < \frac{\arctan(\tan x)}{x} < \frac{\frac{\pi}{2}}{x}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$\lim_{x \to +\infty} \overbrace{\left(1 + \frac{\sin x}{x}\right)^x}^{f(x)} = \text{N.E.} \qquad \left(1 + \frac{1}{a_n}\right)^{a_n} \to e$$

$$a_n = \frac{\pi}{2} + 2n\pi \qquad \boxed{f(a_n)} = \left(1 + \frac{1}{\frac{\pi}{2} + 2n\pi}\right)^{\frac{\pi}{2} + 2n\pi} \to e$$

$$b_n = \frac{3}{2}\pi + 2n\pi \qquad \boxed{f(b_n)} = \left(1 - \frac{1}{\frac{3}{2}\pi + 2n\pi}\right)^{\frac{3}{2}\pi + 2n\pi} \to \frac{1}{e}$$

$$\boxed{\begin{array}{l} a_n \to +\infty \\ b_n \to +\infty \end{array}}$$

$$\lim_{x \to 0} \frac{\sin 3x}{\sqrt{1+x} - 1} =$$

$$= \lim_{x \to 0} \underbrace{\boxed{\frac{\sin 3x}{3x}}}_{\to 1} \cdot \overbrace{\boxed{\frac{x}{\sqrt{1+x} - 1}}}^{\to 2} \cdot \overbrace{\boxed{\frac{3\cancel{x}}{\cancel{x}}}}^{\to 3} = 6$$

$$\boxed{\frac{(1+x)^{\alpha} - 1}{x} \to \alpha}$$

$$\boxed{H} \quad \lim_{x\to 0} \frac{\cos(\sin x)-1}{(e^{3x}-1)\cdot \ln(1+4x)} = \lim_{x\to 0} \frac{-\frac{\cancel{x^2}}{2}}{3\cancel{x}\cdot 4\cancel{x}} = -\frac{1}{24}$$

$\boxed{x\to 0}$

$$e^{3x}-1 \approx 3x$$

$$\ln(1+4x) \approx 4x$$

$$\cos(\sin x)-1 = -\left(1-\cos(\sin x)\right) \approx -\frac{(\sin x)^2}{2} \approx -\frac{x^2}{2}$$

$$1-\cos(f(x)) \approx \frac{(f(x))^2}{2}$$

$$\frac{1-\cos x}{\frac{1}{2}x^2} \to 1$$

$$1-\cos x \approx \frac{1}{2}x^2$$

$$1-\cos(f(x)) \approx \frac{1}{2}(f(x))^2$$

$$\frac{1-\cos(f(x))}{\frac{1}{2}(f(x))^2} \to 1$$

$g(x) = \sin x$ $\qquad f(g(x))$

$f(y) = \boxed{\frac{\cos y - 1}{y^2}} \to -\frac{1}{2}$

$\boxed{H}$ $\displaystyle \lim_{x\to 0} \frac{\cos(\sin x) - 1}{(e^{3x}-1)\cdot \ln(1+4x)} =$

$$= \lim_{x\to 0} \underbrace{\frac{3x}{e^{3x}-1}}_{\downarrow\, 1} \cdot \underbrace{\frac{4x}{\ln(1+4x)}}_{\downarrow\, 1} \cdot \underbrace{\frac{\cos(\sin x)-1}{(\sin x)^2}}_{\downarrow\, -\frac{1}{2}} \cdot \underbrace{\boxed{\frac{(\sin x)^2}{3x\cdot 4x}}}_{\downarrow\, \frac{1}{12}} = -\frac{1}{24}$$

$$\frac{1-\cos x}{x^2} \to \frac{1}{2}$$

$$\lim_{x\to 0} \frac{\cos(\sin x) - 1}{(\sin x)^2} = \lim_{x\to 0} -\frac{1-\cos(\sin x)}{(\sin x)^2} = \lim_{y\to 0} -\frac{1-\cos y}{y^2} = -\frac{1}{2}$$

$y = \sin x$

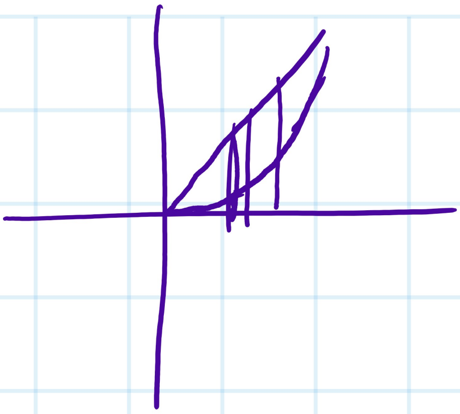
$$\underbrace{\frac{\sin x}{x}}_{\downarrow\, 1} \cdot \underbrace{\frac{\sin x}{x}}_{\downarrow\, 1} \cdot \underbrace{\frac{1}{3\cdot 4}}_{\downarrow\, 12} \to \frac{1}{12}$$

$$\lim_{x\to 0} \frac{\sqrt{1+x^2} - \cos x}{e^{x^2} - e^x} = \lim_{x\to 0} \frac{\sqrt{1+x^2} - 1 + 1 - \cos x}{e^{x^2} - 1 + 1 - e^x} = \lim_{x\to 0} \frac{(\sqrt{1+x^2}-1) + (1-\cos x)}{(e^{x^2}-1) - (e^x - 1)} =$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \qquad e^x - 1 \approx x$$

$$\lim_{x\to 0} \frac{e^{x^2} - 1}{x^2} = 1 \qquad e^{x^2} - 1 \approx x^2$$

$$\lim_{x\to 0} \frac{x^2}{x} = 0$$



$$\lim_{x\to 0} \frac{(\sqrt{1+x^2}-1)+(1-\cos x)}{\underbrace{(e^{x^2}-1)}-\underbrace{(e^x-1)}_{\uparrow}} = \lim_{x\to 0} \frac{}{(e^x-1)\left(\underbrace{\underbrace{\frac{e^{x^2}-1}{e^x-1}}_{\to 0}-1}_{-1}\right)}$$

$$\lim_{x\to 0} \frac{e^{x^2}-1}{e^x-1} = \lim_{x\to 0} \frac{e^{x^2}-1}{x^2}\cdot\frac{x}{e^x-1}\cdot\frac{x^2}{x}$$

$$\boxed{1-\cos x}$$

NO

$$\frac{\tan x - \sin x}{x^3} =$$

$$= \frac{x-x}{x^3} = \frac{0}{x^3} \to 0$$

$$\lim_{x\to 0} \frac{1-\cos x}{\frac{x^2}{2}} = 1 \qquad 1-\cos x \approx \frac{x^2}{2}$$

$$\frac{(1-\cos x)-\frac{x^2}{2}}{1-\cos x} = \underbrace{\frac{1-\cos x}{1-\cos x}}_{\downarrow\, 1} - \underbrace{\frac{\frac{x^2}{2}}{1-\cos x}}_{\downarrow\, 1} \to 1-1=0$$

$$f(x) \approx g(x) \qquad \frac{f(x)-g(x)}{f(x)} = 1 - \underbrace{\frac{g(x)}{f(x)}}_{\downarrow\, 1} \to 0$$

$$\frac{f(x)-g(x)}{g(x)} = \boxed{\underbrace{\boxed{\frac{f(x)-g(x)}{f(x)}}}_{\downarrow\, 0} \cdot \underbrace{\boxed{\frac{f(x)}{g(x)}}}_{\downarrow\, 1}} \to 0$$

$$\frac{(1-\cos x) - \frac{x^2}{2}}{x^2} \to 0$$

$$1-\cos x \approx \frac{x^2}{2} \qquad (1-\cos x) - \frac{x^2}{2} = o(x^2)$$

$$\boxed{1-\cos x = \frac{x^2}{2} + o(x^2)}$$

$$\frac{\sin x}{x} \to 1 \qquad \sin x \approx x \qquad \sin x = x + o(x)$$

$$\tan x - \sin x \approx \frac{x^3}{2} \qquad \tan x - \sin x = \frac{x^3}{2} + o(x^3)$$

$$\frac{e^x-1}{x} \to 1 \qquad e^x - 1 \approx x$$

$$e^x - 1 = x + o(x)$$

$$e^x = 1 + x + o(x)$$

$$\lim_{x\to 0} \frac{\sqrt{1+x^2} - \cos x}{e^{x^2} - e^x} =$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{1+x^2} - 1\right) + (1-\cos x)}{\underbrace{(e^{x^2}-1)} - \underbrace{(e^x - 1)}} = \lim_{x\to 0} \frac{\overbrace{\frac{x^2}{2} + o(x^2)}^{f(x)} + \overbrace{\frac{x^2}{2} + o(x^2)}^{g(x)}}{x^2 + o(x^2) - (x + o(x))}$$

$$\frac{f(x)+g(x)}{x^2} =$$

$$= \lim_{x\to 0} \frac{x^2 + o(x^2)}{\underline{o(x)} + \underline{o(x)} \underline{-x} - \underline{o(x)}} =$$

$$= \lim_{x\to 0} \frac{x^2 + \cancel{o(x^2)}}{-x + \cancel{o(x)}} = \lim_{x\to 0} \frac{x^2\left(1 + \frac{o(x^2)}{x^2}\right)}{x\left(-1 + \frac{o(x)}{x}\right)} = 0$$

$$f(x) \to 0 \qquad \frac{e^{f(x)} - 1}{f(x)} \to 1 \qquad e^{f(x)} - 1 \approx f(x)$$

$$e^{f(x)} - 1 = f(x) + o(f(x))$$

$$\sqrt{1+x} - 1 = \frac{x}{2} + o(x)$$

$$\boxed{\sqrt{1+x^2} - 1 = \frac{x^2}{2} + o(x^2)}$$

$$\boxed{1 - \cos x = \frac{x^2}{2} + o(x^2)}$$

$$\boxed{e^x - 1 = x + o(x)}$$

$$\boxed{e^{x^2} - 1 = x^2 + o(x^2)}$$

$$\frac{\overset{\tan}{\tan x} - \overset{\sin}{\sin x}}{x^3} = \frac{x + o(x) - x - o(x)}{x^3} =$$

$$= \frac{o(x)}{x^3} = ??$$