Lezione 16: Uso degli o-piccoli nei limiti

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2) LIMITI NOTEVOLI IN FORMA DI ASINTOTICA EQUIVALENZA

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$\boxed{C}$ $\lim_{x\to 0} \dfrac{\ln(\cos(\ln(\cos x)))}{\sqrt{\cos x^2}-1}$

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5) LIMITI NOTEVOLI SCRITTI CON GLI O-PICCOLI

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$\boxed{F}$ $\lim_{x\to 0} \dfrac{\ln^2(\sin x+\cos x)-(e^{x+\sin x}-\cos x)^2}{x^2}$

$\boxed{G}$ $\lim_{x\to 0^+} \dfrac{x-\sin x}{x^\alpha}$ $\quad 0<\alpha<3$

$\boxed{H}$ $\lim_{x\to 0} \dfrac{\sin(\sin x)-\sin x}{x^2}$

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7) CONTINUITÀ: DEFINIZIONE E PRIMI ESEMPI (PUNTI DI DISCONTINUITÀ)

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$$\boxed{A}\ \lim_{x\to 0} \frac{\sin(\sin(\sin x))\cdot \overbrace{\ln(\cos x)}^{\ln(1+(\cos x-1))}}{(e^{x\sin x}-1)(\sqrt{1+\tan x}-1)} =$$

$$= \lim_{x\to 0} \frac{\boxed{\frac{\sin(\sin(\sin x))}{\sin(\sin x)}\cdot\frac{\sin(\sin x)}{\sin x}\cdot\frac{\sin x}{x}}\cdot \boxed{x}\cdot\boxed{\frac{\ln(1+(\cos x-1))}{\cos x-1}\cdot\frac{\cos x-1}{-\frac{1}{2}x^2}}\boxed{\left(-\frac{1}{2}x^2\right)}}{\boxed{\frac{e^{x\sin x}-1}{x\cdot\sin x}\cdot\frac{x\cdot\sin x}{x^2}}\cdot\boxed{x^2}\cdot\boxed{\frac{\sqrt{1+\tan x}-1}{\frac{1}{2}\tan x}\cdot\frac{\frac{1}{2}\tan x}{\frac{1}{2}x}}\cdot\boxed{\frac{x}{2}}} =$$

$$= \lim_{x\to} \underbrace{\boxed{}}_{1}\ \underbrace{\boxed{\frac{x\cdot -\frac{1}{2}x^2}{x^2\cdot\frac{x}{2}}}}_{??} = -1$$

PROP. $f(x) \approx F(x)$ per $x\to x_0$ ($f(x)\to 0$, $F(x)\to 0$)

$$\lim_{x\to x_0} \frac{f(x)\cdot g(x)}{h(x)} = \lim_{x\to x_0}\frac{F(x)\cdot g(x)}{h(x)}$$

DIM

$$\lim_{x\to x_0} \boxed{\frac{f(x)\cdot g(x)}{h(x)}} = \lim_{x\to x_0} \underbrace{\boxed{\frac{F(x)\cdot g(x)}{h(x)}}}_{??}\cdot\underbrace{\frac{f(x)}{F(x)}}_{1}$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \sin x \approx x \quad \text{per } x\to 0$$

$$\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2} = 1 \qquad 1-\cos x \approx \frac{x^2}{2} \quad \text{per } x\to 0$$

$$\boxed{\lim_{x\to x_0}\frac{\sin f(x)}{f(x)} \overset{\substack{\text{S.T. } f(x)\to 0 \text{ PER } x\to x_0\\ \downarrow}}{=} \lim_{y\to 0}\frac{\sin y}{y} = 1}$$

SE $f(x)\to 0$ s.t. per $x\to x_0$

$$\sin f(x) \approx f(x) \quad \text{PER } x\to x_0$$

$f(x) \to 0$ PER $x \to x_0$

$\sin f(x) \approx f(x)$ PER $x \to x_0$

$\tan f(x) \approx f(x)$

$1-\cos f(x) \approx \frac{(f(x))^2}{2}$

$\tan(f(x)) - \sin(f(x)) \approx \frac{(f(x))^3}{2}$

$\ln(1+f(x)) \approx f(x)$

$e^{f(x)} - 1 \approx f(x)$

$(1+f(x))^\alpha - 1 \approx \alpha f(x)$

$\boxed{A}$ $$\lim_{x\to 0} \frac{\sin(\sin(\sin x)) \cdot \ln(\cos x)}{(e^{x\sin x}-1)\cdot(\sqrt{1+\tan x}-1)} = \lim_{x\to 0} \frac{x\cdot\left(-\frac{x^2}{2}\right)}{x^2\cdot\frac{x}{2}} = -1$$

$\sin(\sin(\sin x)) \approx \sin(\sin x) \approx \sin x \approx x$

$\ln(\cos x) = \ln(1+(\cos x - 1)) \approx \cos x - 1 = -(1-\cos x) \approx -\frac{x^2}{2}$

$e^{x\sin x} - 1 \approx x\sin x \approx x\cdot x = x^2$

$\sqrt{1+\tan x} - 1 \approx \frac{1}{2}\tan x \approx \frac{x}{2}$

$$\lim_{x\to+\infty} \frac{x^2\cdot\ln\left(\cos\left(\frac{\pi}{2}-\arctan x^2\right)\right)}{\cos\left(\sin\frac{1}{x}\right)-1} = \lim_{x\to+\infty} \frac{\cancel{x^2}\cdot\left(-\frac{1}{2}\cdot\cancel{\frac{1}{x^4}}\right)}{-\frac{1}{2}\cancel{\frac{1}{x^2}}} = 1$$

$$\cos\left(\sin\frac{1}{x}\right)-1 \approx -\frac{1}{2}\left(\sin\frac{1}{x}\right)^2 \approx \boxed{-\frac{1}{2}\left(\frac{1}{x}\right)^2}$$

$$\ln\left(\cos\left(\frac{\pi}{2}-\arctan x^2\right)\right) = \ln\left(1+\left(\cos\left(\frac{\pi}{2}-\arctan x^2\right)-1\right)\right) \approx$$

$$\approx \cos\left(\frac{\pi}{2}-\arctan x^2\right)-1 \approx$$

$$\approx -\frac{1}{2}\left(\frac{\pi}{2}-\arctan x^2\right)^2 =$$

$$= -\frac{1}{2}\left(\arctan\frac{1}{x^2}\right)^2 \approx \boxed{-\frac{1}{2}\left(\frac{1}{x^2}\right)^2}$$

$\boxed{x>0}$

$$\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$$

$$\lim_{x\to 0}\frac{\ln(\cos(\ln(\cos x)))}{\sqrt{\cos x^2}-1} = \lim_{x\to 0}\frac{-\frac{x^4}{8}}{-\frac{x^4}{4}} = \frac{1}{2}$$

$$\sqrt{\cos x^2}-1 = \sqrt{1+(\cos x^2-1)}-1 \approx \frac{1}{2}(\cos x^2-1) \approx \frac{1}{2}\left(-\frac{1}{2}x^4\right)$$

$$= \boxed{-\frac{1}{4}x^4}$$

$$\ln(\cos(\ln(\cos x))) \approx -\frac{(\ln(\cos x))^2}{2} = -\frac{\left(-\frac{x^2}{2}\right)^2}{2} = \boxed{-\frac{x^4}{8}}$$

$$\ln(\cos(x)) = \ln(1+(\cos x-1)) \approx \cos x - 1 \approx -\frac{x^2}{2}$$

$$\ln(\cos f(x)) \approx -\frac{(f(x))^2}{2}$$

$\boxed{D}$ $$\lim_{x\to 0}\frac{A\cdot\sin x^2-(\sin 2x)^2}{x^6}$$ $\boxed{A\in\mathbb{R}}$

$A=1$

$$\lim_{x\to 0}\frac{\sin x^2-(\sin 2x)^2}{x^6} =$$

$$= \lim_{x\to 0}\frac{x^2+o(x^2)-(2x+o(2x))^2}{x^6}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$\sin x \approx x$ per $x\to 0$

$$\frac{(\sin x - x)}{x}\to 0$$

$$\parallel$$

$$\left(\frac{\sin x}{x}-1\right)\to 1-1=0$$

$\sin x = x+o(x)$

$\sin f(x) = f(x)+o(f(x))$

$$\lim_{x\to 0} \frac{\sin x - x}{\sin x} = \lim_{x\to 0}\Big(1-\underbrace{\frac{x}{\sin x}}_{\downarrow\ 1}\Big) = 0$$

$$\sin x - x = o(\sin x)$$

$$\sin x - x = o(x)$$

DEF. DATE $f(x)$, $g(x)$ ENTRAMBE INFINITESIME O INFINITE PER $x\to x_0$ DIREMO CHE SONO DELLO STESSO ORDINE SE

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \ell \in \mathbb{R}-\{0\}$$

OSS. DATE f, g, h INFINITE O INF. PER $x\to x_0$ E TALI CHE $f(x)$ E $g(x)$ HANNO STESSO ORDINE.

ALLORA

$$h(x) = o(f(x)) \iff h(x) = o(g(x))$$

DIM

$$\boxed{h(x) = o(f(x))} \iff \frac{h(x)}{f(x)} \to 0 \iff \frac{h(x)}{g(x)} \cdot \underbrace{\frac{g(x)}{f(x)}}_{\to\ \ell \neq 0}$$

$$\boxed{h(x) = o(g(x))} \iff \frac{h(x)}{g(x)} \to 0$$

OSS. SE $f(x)$ E $g(x)$ PER $x\to x_0$ SONO ENTRAMBE INF. O INF.-SIME e $f(x) \approx g(x)$ PER $x\to x_0$ ALLORA

$$f(x) - g(x) = o(f(x)) = o(g(x))$$

DIM

$$\lim_{x\to x_0}\frac{f(x)-g(x)}{f(x)} = \lim_{x\to x_0}\Big(1 - \underbrace{\frac{g(x)}{f(x)}}_{\downarrow\ 1}\Big) = 0$$

PER $x \to 0$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \iff \sin x \simeq x \iff \sin x - x = o(x)$$

$$\sin x = x + o(x) \qquad \alpha < 3$$

$$\sin x = x - \frac{x^3}{6} + o(x^\alpha) \quad (\alpha < 5) \qquad x + o(x^\alpha)$$

OSS. DATE f, g, h (INF.) PER $x \to x_0$ È TALI CHE

$$f(x) = o(g(x)) \quad g(x) = o(h(x))$$

ALLORA

$$f(x) = o(h(x))$$

DIM.

$$\left(\frac{f(x)}{g(x)} \to 0\right) \text{ E } \left(\frac{g(x)}{h(x)} \to 0\right) \Rightarrow \frac{f(x)}{h(x)} \to 0$$

$$\frac{f(x)}{h(x)} = \underbrace{\frac{f(x)}{g(x)}}_{\to 0} \cdot \underbrace{\frac{g(x)}{h(x)}}_{\to 0}$$

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \boxed{\lim_{x\to 0} \frac{x - x}{x^3} = \lim_{x\to 0} \frac{0}{x^3} = 0}$$ NO!!

SI →

$$= \lim_{x\to 0} \frac{(x + o(x)) - (x + o(x))}{x^3} =$$

$$= \lim_{x\to 0} \frac{o(x)}{x^3} = \dots ??$$

$$\frac{\tan x}{x} \to 1$$

$$\tan x \simeq x$$

$$\tan x - x = o(x)$$

$$\tan x = x + o(x)$$

OSS. DATE f, g, h (INF.) PER $x \to x_0$ E TALI CHE

$$f(x) = o(h(x)) \quad g(x) = o(h(x))$$

ALLORA $\underline{\alpha f(x) + \beta g(x)} = o(h(x))$

DIM

$$\frac{\alpha f(x) + \beta g(x)}{h(x)} \to 0 \ ?$$

$$\parallel$$

$$\left(\alpha \underbrace{\frac{f(x)}{h(x)}}_{\downarrow\, 0} + \beta \underbrace{\frac{g(x)}{h(x)}}_{\downarrow\, 0}\right) \to \alpha 0 + \beta 0 = 0$$

: $\boxed{D}$ $\lim_{x\to 0} \frac{A\cdot \sin x^2 - (\sin 2x)^2}{x^6}$ $\boxed{A\in\mathbb{R}}$

$A=1$

$(o(f))^2 = o(f^2)$

$g\cdot o(f) = o(f\cdot g)$

$$\lim_{x\to 0} \frac{\sin x^2 - (\sin 2x)^2}{x^6} =$$

$$= \lim_{x\to 0} \frac{x^2 + o(x^2) - (2x + o(2x))^2}{x^6} =$$

$$= \lim_{x\to 0} \frac{x^2 + o(x^2) - (2x)^2 - (o(2x))^2 - 2\cdot 2x\cdot o(2x)}{x^6} =$$

$$= \lim_{x\to 0} \frac{x^2 + o(x^2) - 4x^2 - o(x^2) - o(x^2)}{x^6} =$$

$$= \lim_{x\to 0} \frac{-3x^2 + o(x^2)}{x^6} = \lim_{x\to 0} \frac{-3\cancel{x^2}\overbrace{\left(1 - \frac{o(x^2)}{3x^2}\right)}^{\to 1}}{x^4} = -\infty$$

$$\lim_{x\to 0} \frac{4\sin x^2 - (\sin 2x)^2}{x^6} = \lim_{x\to 0} \frac{4(x^2 + o(x^2)) - ((2x) + o(x))^2}{x^6} =$$

$$= \lim_{x\to 0} \frac{\cancel{4x^2} + o(x^2) - \cancel{4x^2} - o(x^2) - o(x^2)}{x^6} =$$

$$= \lim_{x\to 0} \frac{o(x^2)'}{x^6} = ???$$

$$\lim_{x\to 0} \frac{4\sin x^2 - 4\sin^2 x\cos^2 x}{x^6} = \lim_{x\to 0} \frac{4\sin(x^2) - \cancel{4}\sin^2 x + 4\sin^2 x - 4\sin^2 x\cos^2 x}{x^6} =$$

$$= \lim_{x\to 0} \frac{4(\sin x^2 - \sin^2 x)}{x^6} + \frac{4\sin^2 x(1-\cos^2 x)}{x^6}$$

$(\tan x - \sin x)$

$$\lim_{x\to 0^+} \frac{x-\sin x}{x^2} =$$

$0 < x < \frac{\pi}{2}$

$0 < \sin x < x < \tan x$

$$0 < \underbrace{\frac{x-\sin x}{x^2}}_{\downarrow\, 0} < \underbrace{\frac{\tan x - \sin x}{x^3}}_{\downarrow\, \frac{1}{2}} \cdot \underset{\downarrow\, 0}{x}$$

$\boxed{0<\alpha<3}$

$$0 < \frac{x-\sin x}{x^\alpha} < \frac{\tan x - \sin x}{x^\alpha} = \underbrace{\frac{\tan x - \sin x}{x^3}}_{\downarrow\, \frac{1}{2}} \cdot \underbrace{x^{3-\alpha}}_{\downarrow\, 0}$$

$x - \sin x = o(x^\alpha)$ $\boxed{\boxed{\alpha<3}}$

$\underline{\sin x - x = o(x^\alpha)}$ $\boxed{\tan x = x + o(x^\alpha)}$

$\sin x = \boxed{x + o(x^\alpha)}$ $\boxed{\alpha<3}$ $\frac{\tan x - x}{x^\alpha} \to 0$

$$0 < \underbrace{\frac{\tan x - x}{x^\alpha}}_{\downarrow\, 0} < \boxed{\underbrace{\frac{\tan x - \sin x}{x^\alpha}}_{\downarrow\, 0}}$$

$$\lim_{x\to 0} \frac{4\sin(x^2) - 4\sin^2 x \sin^2}{x^4} =$$

$$= \lim_{x\to 0} \frac{4\left(x^2 + o(x^{2\alpha})\right) - 4\left(x + o(x^\alpha)\right)^2 (1 - \sin^2 x)}{x^4} =$$

$$= \lim_{x\to 0} \frac{\cancel{4x^2} + o(x^{2\alpha}) - (\cancel{4x^2} + 4\,o(x^{2\alpha}) + 8\,o(x^{\alpha+1})) + \sin^2 x(\qquad)}{x^4} =$$

$$= \lim_{x\to 0} \frac{\qquad}{}$$