Lezione 17: Funzioni Continue (I)

INDICE

DA SCORSA LEZIONE

6) ESEMPI:

$\boxed{E}$ $$\lim_{x\to 0} \frac{(x-\sin 2x)^2 + \cos x - e^{x^2}}{(\tan x + \sin x)\cdot \ln(1+\sin x)}$$

$\boxed{F}$ $$\lim_{x\to 0} \frac{\ln^2(\sin x + \cos x) - (e^{x+\sin x} - \cos x)^2}{x^2}$$

$\boxed{G}$ $$\lim_{x\to 0^+} \frac{x - \sin x}{x^\alpha} \qquad 0<\alpha<3$$

$\boxed{H}$ $$\lim_{x\to 0} \frac{\sin(\sin x) - \sin x}{x^2}$$

$$\lim_{x\to 0} \frac{(x-\sin 2x)^2 + \cos x - e^{x^2}}{(\tan x + \sin x)\cdot \ln(1+\sin x)} =$$

$$= \lim_{x\to 0} \frac{\left(x-(2x+o(x))\right)^2 + \cancel{1} - \frac{x^2}{2} + o(x^2) - \left(\cancel{1} + x^2 + o(x^2)\right)}{\left(x+o(x)+x+o(x)\right)\cdot\left(x+o(x)\right)} =$$

$$\ln(1+\sin x) = \underline{\overbrace{\sin x}} + \overbrace{o(\sin x)}^{o(x)} = x + o(x) + o(x) = x + o(x)$$

$$= \lim_{x\to 0} \frac{\left(-x+o(x)\right)^2 - \frac{3}{2}x^2 + o(x^2)}{\left(2x+o(x)\right)\left(x+o(x)\right)} =$$

$$= \lim_{x\to 0} \frac{x^2 - 2x\,o(x) + (o(x))^2 - \frac{3}{2}x^2 + o(x^2)}{2x^2 + \underbrace{2x\,o(x)} + \underbrace{x\,o(x)} + (o(x))^2} =$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2}x^2 + \overbrace{o(x^2) + o(x^2) + o(x^2)}}{2x^2 + o(x^2) + o(x^2) + o(x^2)} =$$

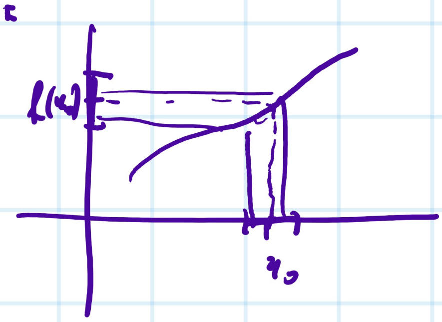
$$= \lim_{x\to 0} \frac{\overbrace{-\frac{1}{2}x^2} + \overbrace{o(x^2)}}{\underbrace{2x^2} + \underbrace{o(x^2)}} = \lim_{x\to 0} \frac{-\frac{1}{2}\cancel{x^2}\overbrace{\left(1 + \boxed{\frac{o(x^2)}{-\frac{1}{2}x^2}}^{\,0}\right)}^{1}}{2\cancel{x^2}\underbrace{\left(1 + \boxed{\frac{o(x^2)}{2x^2}}_{\,0}\right)}_{1}} = -\frac{1}{4}$$

F. CONTINUE

DEF Dato $A \subset \mathbb{R}$, $f: A \to \mathbb{R}$ e $x_0 \in A$ diremo che f è continua in x_0 se

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ t.c. } \left(\underline{|x - x_0| < \delta} \text{ e } \underline{x \in A} \right) \Rightarrow \underline{|f(x) - f(x_0)| < \varepsilon}$$

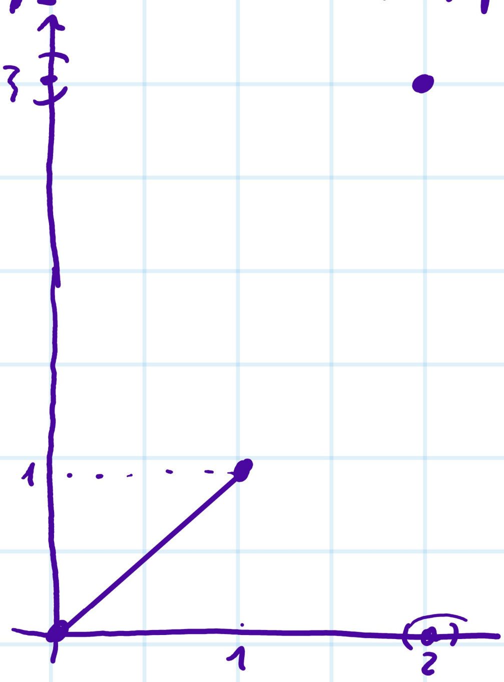
Se f è continua su tutti gli $x \in A$ si dice che f è continua su A.



OSS

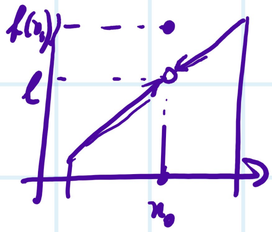
f è cont. su tutti i punti isolati del dominio.

Es. $f: [0,1] \cup \{2\} \to \mathbb{R}$ t.c. $f(x) = \begin{cases} x & \text{se } x \in [0,1] \\ 3 & \text{se } x = 2 \end{cases}$



DEF. Dati $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ e $x_0 \in A$, se f non è continua in x_0, diremo che x_0 è p.t. di discontinuità di tipo:

1) ELIMINABILE se $\lim_{x \to x_0} f(x) = \ell \neq f(x_0)$

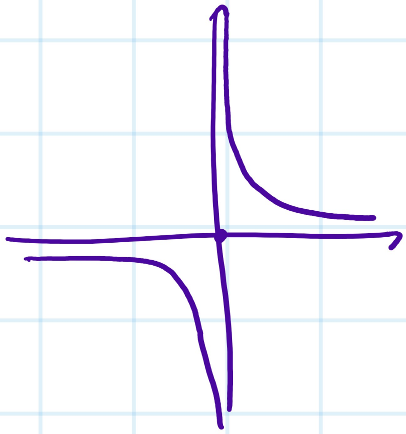


2) DI TIPO SALTO se $\exists^{\text{no}}$ finiti $\lim_{x \to x_0^+} f(x)$ e $\lim_{x \to x_0^-} f(x)$ e sono diversi.

3) [ALTRO]

ES. $\boxed{f(x) = \frac{1}{x}}$

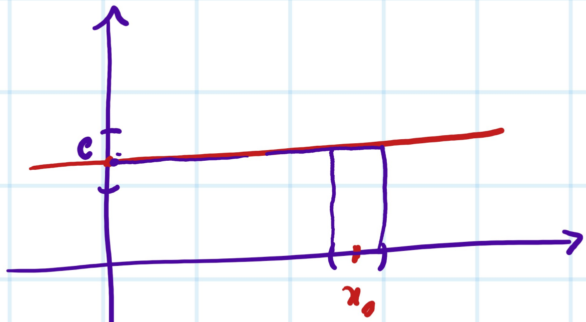
$$f: \mathbb{R} - \{0\} \to \mathbb{R}$$
$$x \mapsto \frac{1}{x}$$



ES. STUPIDI

$$f: A \to \mathbb{R}$$
$$x \mapsto c$$

è cont.



$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x$$

è cont.

$$|x - x_0|$$

$\underline{\forall \varepsilon > 0}$ $\underline{\exists \delta > 0}$ t.c. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$$x^\alpha = x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

$$x^\alpha = e^{\ln(x^\alpha)} = e^{\boxed{\alpha \boxed{\ln x}}}$$

$\boxed{\text{T. PONTE}}$ DATI $A \subset \mathbb{R}$, $f: A \to \mathbb{R}$ E $x_0 \in A$, È EQUIVALENTE DIRE CHE:

1) f È CONTINUA IN x_0

2) $\forall (a_n)$ A VALORI IN A t.c. $a_n \to x_0$ SI HA $f(a_n) \to f(x_0)$

$\boxed{\text{DIM}}$ $(1) \Rightarrow (2)$

SO CHE: $\left[\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ t.c. } \left.\begin{array}{l} |x - x_0| < \delta \\ x \in A \end{array}\right\} \Rightarrow |f(x) - f(x_0)| < \varepsilon \right.$

DEVO MOSTRARE CHE:

$\left[\begin{array}{l} \forall \varepsilon > 0, \text{ DEF IN } n, \ |f(a_n) - f(x_0)| < \varepsilon \\ \text{SAPENDO CHE } \underbrace{a_n \to x_0}. \end{array} \right.$

$\Downarrow$

DEF. IN n $|a_n - x_0| < \delta$

$\Downarrow$

DEF. IN n, $|f(a_n) - f(x_0)| < \varepsilon$

$(2) \Rightarrow (1)$ P.A. $\exists \varepsilon > 0$ t.c. $\forall \delta > 0$ $\exists x \in A$ t.c. $|x - x_0| < \delta$
MA $|f(x) - f(x_0)| > \varepsilon$

PRENDO $\delta = \frac{1}{n}$ $n \in \mathbb{N} \setminus \{0\}$

$\delta_1 = 1$

$\delta_2 = \frac{1}{2}$

$\vdots$

$\delta_n = \frac{1}{n}$

$\vdots$

PER OGNI $\delta_n = \frac{1}{n}$ SO CHE $\exists x \in A$ t.c. $|x - x_0| < \frac{1}{n}$ MA $|f(x) - f(x_0)| \geq \varepsilon$

LO CHIAMO a_n

$|a_n - x_0| < \frac{1}{n}$ $\quad |f(a_n) - f(x_0)| \geq \varepsilon$

PRENDO (a_n), TALE $a_n \to x_0$ MA $f(a_n) \not\to f(x_0)$

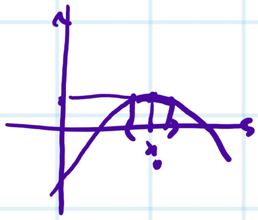
(ASSURDO PERCHÉ VALE (2))

QUINDI VALE (1)

$\boxed{\text{T. PERM. SEGNO}}$ DATI $A \subset \mathbb{R}$, $f: A \to \mathbb{R}$, $x_0 \in A$ t.c. f CONT. IN x_0, CON $f(x_0) > 0$

ALLORA $\exists \delta > 0$ t.c. $\forall x \in (x_0 - \delta, x_0 + \delta) \cap A$ SI HA $f(x) > 0$

$\boxed{\text{DIM}}$ $\forall \varepsilon > 0\ \exists \delta > 0$ t.c. $\left.\begin{matrix} |x - x_0| < \delta \\ x \in A \end{matrix}\right\} \Rightarrow f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$



IN PARTICOLARE SIA $\varepsilon = \frac{f(x_0)}{2}$ ALLORA $\exists \delta > 0$ t.c.

$\forall x \in A \cap (x_0 - \delta, x_0 + \delta)$ SI HA $\boxed{f(x_0) - \frac{1}{2} f(x_0) < f(x)} < f(x_0) + \frac{1}{2} f(x_0)$

$0 < \frac{1}{2} f(x_0) < f(x)$

$\boxed{\text{T.}}$ DATI $A \subset \mathbb{R}$, $x_0 \in A$ E $f, g : A \to \mathbb{R}$ CONTINUE IN x_0. ALLORA:

1) $f + g$ È CONTINUA IN x_0

2) $f \cdot g$ È CONTINUA IN x_0

3) SE $g(x_0) \neq 0$ ALLORA $\frac{f}{g}$ È CONTINUA IN x_0

$\boxed{\text{DIM}}$

1) DEVO MOSTRARE CHE

$$\boxed{\forall \varepsilon>0 \ \exists \delta>0 \ \text{t.c.} \ \left.\begin{array}{l}|x-x_0|<\delta \\ x\in A\end{array}\right\} \Rightarrow \left|\left(f(x)+g(x)\right)-\left(f(x_0)+g(x_0)\right)\right|<\varepsilon}$$

PERCHÉ f È CONT.: $\forall \varepsilon>0$ PRENDO $\delta_1>0$ t.c. $\left.\begin{array}{l}|x-x_0|<\delta_1 \\ x\in A\end{array}\right\} \Rightarrow |f(x)-f(x_0)|<\frac{\varepsilon}{2}$

PERCHÉ g È CONT.: PRENDO $\delta_2>0$ t.c. $\left.\begin{array}{l}|x-x_0|<\delta_2 \\ x\in A\end{array}\right\} \Rightarrow |g(x)-g(x_0)|<\frac{\varepsilon}{2}$

PRENDO $\delta=\min\{\delta_1,\delta_2\}$, SE $\left.\begin{array}{l}|x-x_0|<\delta \\ x\in A\end{array}\right\} \Rightarrow \blacksquare$

QUINDI

$$\left|f(x)+g(x)-\left(f(x_0)+g(x_0)\right)\right| = \left|\left(f(x)-f(x_0)\right)+\left(g(x)-g(x_0)\right)\right| \leq$$

$$\leq |f(x)-f(x_0)|+|g(x)-g(x_0)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

OSS. DATI $A\subseteq\mathbb{R}$, $x_0\in A$, $f:A\to\mathbb{R}$ CONT. IN x_0, ALLORA $\exists$ UN INTORNO DI x_0 IN CUI f È LIMITATA.

DIMO BASTA APPLICARE LA DEF. DI CONTINUITÀ IN x_0 CON $\varepsilon=1$.

(2) DEVO DIMOSTRARE CHE $\forall \varepsilon>0 \ \exists \delta>0$ t.c. $\left.\begin{array}{l}|x-x_0|<\delta \\ x\in A\end{array}\right\} \Rightarrow \underline{|f(x)g(x)-f(x_0)g(x_0)|<\varepsilon}$

$$|f(x)g(x)-f(x_0)g(x_0)| = |\underline{f(x)g(x)-f(x)g(x_0)}+\underline{f(x)g(x_0)-f(x_0)g(x_0)}| \leq$$

$$\leq \underline{|f(x)|}\cdot|g(x)-g(x_0)|+\underline{|g(x_0)|}\cdot\underline{|f(x)-f(x_0)|} \leq \ldots$$

$\forall \varepsilon > 0$ PRENDO $\delta_0 > 0$ t.c. $\left.\begin{array}{l}|x - x_0| < \delta_0 \\ x \in A\end{array}\right\} \Rightarrow |f(x)| < |f(x_0)| + 1$

$\rightarrow$ $\delta_1 > 0$ t.c. $\left.\begin{array}{l}|x - x_0| < \delta_1 \\ x \in A\end{array}\right\} \Rightarrow |f(x) - f(x_0)| < \dfrac{\varepsilon}{2(|g(x_0)|+1)}$

$\rightarrow$ $\delta_2 > 0$ t.c. $\left.\begin{array}{l}|x - x_0| < \delta_2 \\ x \in A\end{array}\right\} \Rightarrow |g(x) - g(x_0)| < \dfrac{\varepsilon}{2(|f(x_0)|+1)}$

PRENDO $\delta = \text{MIN}\{\delta_0, \delta_1, \delta_2\}$, $\left.\begin{array}{l}|x - x_0| < \delta \\ x \in A\end{array}\right\} \Rightarrow$

$$\ldots \leq \underbrace{|f(x)|}\cdot|g(x) - g(x_0)| + \underbrace{|g(x_0)|}\cdot\underbrace{|f(x) - f(x_0)|} \leq$$

$$\leq \cancel{(|f(x_0)|+1)} \cdot \frac{\varepsilon}{2\cancel{(|f(x_0)|+1)}} + \boxed{|g(x_0)|} \cdot \frac{\varepsilon}{2\boxed{(|g(x_0)|+1)}} <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

[...]

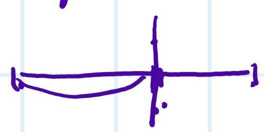
$\boxed{\text{T. ZERI}}$ DATI $I = [a,b]$, $f: I \to \mathbb{R}$ CONTINUA E TALE CHE $f(a) \cdot f(b) < 0$

ALLORA $\exists c \in (a,b)$ t.c. $f(c) = 0$.

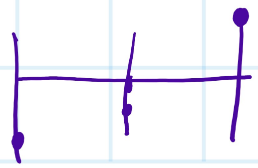
SUPPONIAMO $f(a) < 0$ $f(b) > 0$

$\boxed{\text{DIM}}$ SIA $I_0 = I = [a,b] = [a_0, b_0]$

PRENDO $f\left(\frac{a_0+b_0}{2}\right) = \begin{cases} 0 \\ < 0 \\ > 0 \end{cases}$



SE $f\left(\frac{a_0+b_0}{2}\right) = 0$ HO VINTO



$$\begin{cases} \text{SE } f\left(\frac{a_0+b_0}{2}\right) > 0 & \text{PONGO } a_1 = a_0 \quad b_1 = \frac{a_0+b_0}{2} \\ \text{SE } f\left(\frac{a_0+b_0}{2}\right) < 0 & \text{PONGO } a_1 = \frac{a_0+b_0}{2} \quad b_1 = b_0 \end{cases}$$

$$I_1 = [a_1, b_1] \quad \text{t.c.} \quad (b_1 - a_1) = \frac{1}{2}(b_0 - a_0)$$

E $f(a_1) < 0$ $f(b_1) > 0$

VADO AVANTI

$$I_0 = [a_0, b_0] \qquad f(a_0) < 0 \quad f(b_0) > 0$$

$$I_1 = [a_1, b_1] \qquad f(a_1) < 0 \quad f(b_1) > 0 \quad (b_1 - a_1) = \frac{1}{2}(b_0 - a_0)$$

$$\vdots$$

$$I_n = [a_n, b_n] \qquad f(a_n) < 0 \quad f(b_n) > 0 \quad (b_n - a_n) = \frac{1}{2}(b_{n-1} - a_{n-1}) = \dots = \frac{1}{2^n}(b_0 - a_0)$$

$$\vdots$$

CONSIDERO (a_n), (b_n) LIMITATE, E MONOTONE (a_n CRESC., b_n DECR.)

QUINDI $\exists^{no}$ $l_1, l_2 \in \mathbb{R}$ t.c. $a_n \to l_1$ E $b_n \to l_2$.

MA SI HA $l_1 = l_2$ PERCHÈ:

$$\left.\underbrace{\underset{\downarrow \, l_2}{b_n} - \underset{\downarrow \, l_1}{a_n}}_{l_2 - l_1} = \frac{b_0 - a_0}{2^n} \to 0 \right\} \Rightarrow 0 = l_2 - l_1 \Rightarrow l_2 = l_1 = c$$

$$a_n \to c \Rightarrow \overline{f(a_n)} \to f(c)$$

$$b_n \to \quad \Rightarrow \overline{f(b_n)} \to f(c)$$

$$\boxed{c \in [a,b]}$$

$$\begin{aligned} f(a_n) < 0 &\Rightarrow f(c) \leq 0 \\ f(b_n) > 0 &\Rightarrow f(c) \geq 0 \end{aligned} \Rightarrow \boxed{f(c) = 0}$$

MA $c \neq a$
$c \neq b$
PERCHÈ $f(a) \neq 0$
$f(b) \neq 0$