Lezione 19: Esempio di I esonero

1 TROVARE (a_n) CON LE TUTTE SEGUENTI PROPRIETÀ

1) FREQ. IN n, $(\forall \varepsilon > 0 \quad |a_n - 5| < \varepsilon)$
2) DEF. IN n, $|a_{n+1} - a_n| < 2$
3) $\forall M \in \mathbb{R}$, FREQ. IN n, $a_n > M$

2 CONFRONTARE GLI ORDINI DI INFINITO DI: $a_n = (n!)^{n+1+\frac{1}{\log_2 n}}$, $b_n = ((n+1)!)^n$, $c_n = (n!)^{n+1} \cdot n^{2023}$

3 TROVARE $\liminf(a_n)$ E $\limsup(a_n)$ CON $a_n = \left(1 + \frac{\sin\left(\frac{\pi n}{3}\right)}{n}\right)^n$

4 CALCOLARE
$$\lim_{n \to +\infty} \frac{\ln\left(1 + \frac{1}{(2n)!}\right) \ln\left(1 + e^{2^n}\right)}{\sin\left(\frac{1}{n^{2n}}\right) \cdot \cos(\pi n!)}$$

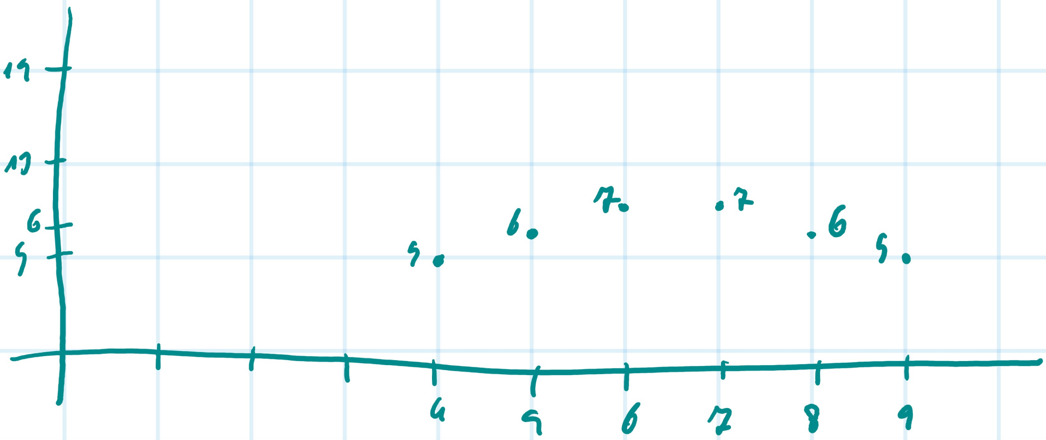
5 CALCOLARE
$$\lim_{x \to +\infty} \left| \frac{\cos(1000 \cdot \pi \sin x) + \cos(2023 \cdot \sin x)}{2} \right|^x$$

6 SIA $f: \mathbb{R} \to \mathbb{R}$ UNA FUNZIONE CONTINUA E PERIODICA SIA DI PERIODO 1 SIA DI PERIODO $\sqrt{2}$.
MOSTRARE CHE f È COSTANTE.

SOLUZIONI

1 TROVARE (a_n) CON LE TUTTE SEGUENTI PROPRIETÀ

1) FREQ. IN n, $(\forall \varepsilon > 0 \quad |a_n - 5| < \varepsilon)$ $\iff$ (a_n) HA SSUCC. $(a_{n_k}) \equiv 5$
2) DEF. IN n, $|a_{n+1} - a_n| < 2$
3) $\forall M \in \mathbb{R}$, FREQ. IN n, $a_n > M$



$A = \{k^2 \mid k \in \mathbb{N}\}$

$$a_n = \begin{cases} 5 & \text{SE } n = k^2 \\ 5 + d(n, A) \end{cases}$$

$$\big| d(n, A) - d(n+1, A) \big| \leq 1$$

$a_{n_k} = 5 + d(n_k, A) =$ $\qquad n = k^2 + k$

$= 5 + d(k^2 + k, k^2) = 5 + k$ $\qquad d(n, k^2) = k$

$\uparrow$

$n_k = \boxed{k^2 + k}$ $\qquad d(n, (k+1)^2) = k + 1$

$$\boxed{c_n \ll a_n \ll b_n} \qquad c_n = o(a_n) \quad a_n = o(b_n) \quad c_n = o(b_n)$$

2 CONFRONTARE GLI ORDINI DI INFINITO DI: $a_n = (n!)^{n+1+\frac{1}{\log_2 n}}$, $b_n = ((n+1)!)^n$, $c_n = (n!)^{n+1} \cdot n^{2023}$

$$\frac{a_n}{c_n} = \frac{(n!)^{n+1+\frac{1}{\log_2 n}}}{(n!)^{n+1} \cdot n^{2023}} = \frac{\cancel{(n!)^{n+1}} \cdot (n!)^{\frac{1}{\log_2 n}}}{\cancel{(n!)^{n+1}} \cdot n^{2023}} = \qquad \underbrace{n^{\alpha n}}_{e^n} \ll n! \ll n^n$$

$$= \frac{e^{\ln\left((n!)^{\frac{1}{\log_2 n}}\right)}}{n^{2023}} = \frac{e^{\frac{\ln 2}{\ln n} \cdot \ln(n!)}}{n^{2023}} >$$

$$> \frac{e^{\frac{\ln 2}{\ln n} \cdot \ln(e^n)}}{n^{2023}} = \frac{2^{\frac{1}{\ln n} \cdot n}}{n^{2023}} =$$

$$= \frac{2^{\boxed{\frac{\sqrt{n}}{\ln n}} \cdot \sqrt{n}}}{n^{2023}} > \frac{2^{\sqrt{n}}}{n^{2023}} = \frac{2^{\sqrt{n}}}{(\sqrt{n})^{4046}} \longrightarrow +\infty$$

$$\frac{b_n}{c_n} = \frac{((n+1)!)^n}{(n!)^{n+1} \cdot n^{2023}} = \left(\frac{(n+1)\cancel{!}}{\cancel{n!}}\right)^n \cdot \frac{1}{n! \cdot n^{2023}} = \boxed{\frac{(n+1)^n}{\underbrace{n! \cdot n^{2023}}_{(n+2023)!}}} > \quad (*_n)$$

$$> \frac{(n+1)^n}{(n+2023)!} \longrightarrow +\infty$$

$$n! \cdot n^{2023} = \overbrace{n \cdot n \cdot n \cdot \ldots n}^{2023} \cdot n(n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

$$(n+2023)! \quad \underbrace{(n+2023) \ldots (n+2)(n+1)}_{2023} \, n(n-1) \cdot \ldots \ldots \; 3 \cdot 2 \cdot 1$$

$$A_n = \frac{(n+1)^n}{n!\cdot n^{2023}}$$

$$\frac{A_{n+1}}{A_n} = \frac{(n+2)^{n+1}}{(n+1)\cancel{!}\cdot(n+1)^{2023}} \cdot \frac{\cancel{n!}\cdot n^{2023}}{(n+1)^n} = \frac{(n+2)}{n+1}\cdot\left(\frac{(n+2)}{(n+1)}\right)^n \cdot \left(\frac{n}{n+1}\right)^{2023} =$$

$$= \boxed{\underbrace{\left(1+\frac{1}{n+1}\right)^{n+1}}_{\downarrow\, e}\cdot \underbrace{\left(1-\frac{1}{n+1}\right)^{2023}}_{\downarrow\, 1}} \to e$$

MODO ALTERNATIVO

$$\frac{a_n}{b_n} = \frac{(n!)^{n+1+\frac{1}{\log_2 n}}}{((n+1)!)^n} = \frac{(n!)^n \cdot n! \cdot (n!)^{\frac{1}{\log_2 n}}}{((n+1)!)^n} =$$

$$= \left(\frac{\cancel{n!}}{(n+1)\cancel{!}}\right)^n \cdot n! \cdot e^{\frac{\ln 2}{\ln n}\cdot \ln(n!)} =$$

$$= \frac{n!\cdot 2^{\frac{\ln(n!)}{\ln n}}}{(n+1)^n} < \frac{n!\cdot 2^{\frac{\ln(n^n)}{\ln n}}}{(n+1)^n} =$$

$$= \boxed{\frac{n!\cdot 2^n}{(n+1)^n}} \to 0$$

B_n

$$\frac{B_{n+1}}{B_n} = \frac{(n+1)\cancel{!}\cdot 2^{n+1}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{\cancel{n!}\cdot\cancel{2^n}} = 2\cdot\frac{(n+1)^{n+1}}{(n+2)^{n+1}} = 2\cdot\left(\frac{n+1}{n+2}\right)^{n+1} = \frac{2}{\left(\frac{n+2}{n+1}\right)^{n+1}} = \frac{2}{\left(\frac{n+1+1}{n+1}\right)^{n+1}} = \frac{2}{\left(1+\frac{1}{n+1}\right)^{n+1}} \to \frac{2}{e} < 1$$

$\boxed{3}$ TROVARE LIMINF(a_n) E LIMSUP(a_n) CON $a_n = \left(1 + \frac{\sin\left(\frac{\pi n}{3}\right)}{n}\right)^n$

$\boxed{n_k = 6k}$

$\boxed{n_k = 1+6k}$ $\longmapsto$ $a_{n_k} = \left(1 + \frac{\sin\left(\frac{1+6k}{3}\pi\right)}{1+6k}\right)^{1+6k} = \left(1 + \frac{\frac{\sqrt{3}}{2}}{1+6k}\right)^{1+6k} \to \boxed{e^{\frac{\sqrt{3}}{2}}}$

$n_k = 2+6k$

$\boxed{n_k = 3+6k}$

$\boxed{\begin{matrix} n_k = 4+6k \\ n_k = 5+6k \end{matrix}}$ $\longrightarrow$ $a_{n_k} = \left(1 + \frac{-\frac{\sqrt{3}}{2}}{4+6k}\right)^{4+6k} \to e^{-\frac{\sqrt{3}}{2}} = \boxed{\frac{1}{e^{\frac{\sqrt{3}}{2}}}}$

$$\ln\left(e^{2^n}\cdot\left(\frac{1}{e^{2^n}}+1\right)\right)$$

$\boxed{4}$ CALCOLARE $\displaystyle\lim_{n\to+\infty}\frac{\ln\left(1+\frac{1}{(2n)!}\right)\cdot\ln\left(1+e^{2^n}\right)}{\sin\left(\frac{1}{n^{2n}}\right)\cdot\underbrace{\cos(\pi n!)}_{1}} =$

$a_n \to 0$

$\dfrac{\ln(1+a_n)}{a_n}\to 1$

$$= \lim_{n\to+\infty}\frac{\underbrace{\boxed{\dfrac{\ln\left(1+\frac{1}{(2n)!}\right)}{\frac{1}{(2n)!}}}}_{\to 1}\cdot\frac{1}{(2n)!}\cdot\underbrace{\boxed{\dfrac{\left(2^n+\ln\left(1+\frac{1}{e^{2^n}}\right)\right)}{2^n}}}_{\to 1}\cdot 2^n}{\underbrace{\boxed{\dfrac{\sin\left(\frac{1}{n^{2n}}\right)}{\frac{1}{n^{2n}}}}}_{\to 1}\cdot\frac{1}{n^{2n}}} = +\infty$$

$+\infty$

$$\frac{n^{2n}\cdot 2^n}{(2n)!} = \frac{n^n\cdot \overbrace{n^n\cdot 2^n}^{(2n)^n}}{(2n)!} = \frac{(2n)^n\cdot n^n}{(2n)!} =$$

$$= \frac{\overbrace{(2n)\cdot(2n)\cdot\ldots\cdot(2n)}^{n}\;\overbrace{n\cdot n\cdot\ldots\cdot n}^{n}}{\underbrace{(2n)\cdot(2n-1)\cdot\ldots\cdot(n+1)}_{n}\;\underbrace{n(n-1)\cdot\ldots\cdot 3\cdot 2\cdot 1}_{n}} \geq$$

$$\geq 1\cdot\boxed{\frac{n}{1}} = n \to +\infty$$

5

CALCOLARE $\boxed{\lim_{x\to+\infty} \left| \frac{\cos(1000\cdot\pi \sin x) + \cos(2023\cdot \sin x)}{2} \right|^x} = 0$

(?) $\leq k < 1$

$f(\sin x)$

$f(\sin(x+2k\pi))$

$0 \leq \bullet \leq k^x$

$\downarrow \quad \downarrow \quad \downarrow$

$0 \quad 0 \quad 0$

$$\begin{cases} 1000\pi\cdot y = 2m\pi \\ 2023\cdot y = 2n\pi \end{cases}$$

$y = \boxed{\frac{2m}{1000}}$

$y = \frac{2n\pi}{2023}$

$\frac{2m}{1000} = \frac{2n}{2023}\cdot\pi$

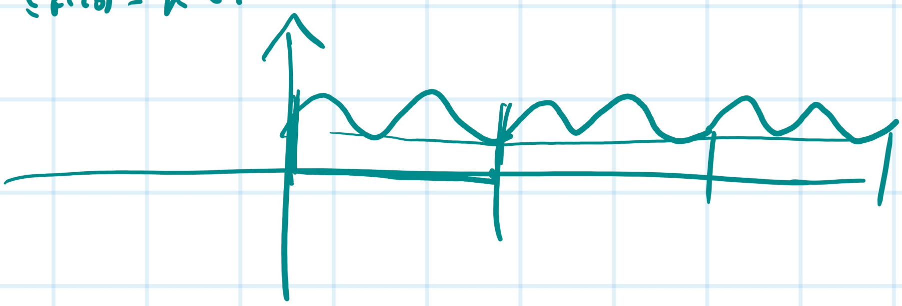
$\pi = \boxed{\frac{2m}{1000}\cdot\frac{2023}{2n}}$

$x_0 \in [0, 2\pi]$ SIA PT. DI MAX

PER $f(x) \left| \frac{\cos(\dots)+\cos(\dots)}{2} \right|$

$\forall x \in \mathbb{R}$

$f(x) \leq f(x_0) = k < 1$



CALCOLARE $\lim_{x\to+\infty}\left|\frac{\cos(1000\cdot\pi\sin x)+\cos(2023\cdot\sin x)}{2}\right|^{x}=(f(x))^{x}$

$0\le f(x)\le k<1$

$\uparrow$

$0\le (f(x))^{x}\le k^{x}$

$f(x)<1$

$$\begin{cases}1000\cancel{\pi}\sin x = 2n\cancel{\pi}\\ 2023\sin x = 2m\pi\end{cases}$$

$\sin x = \boxed{\frac{n}{500}}$

$\sin x = \boxed{\frac{2m}{2023}\cdot\pi}$

$$g(y)=\left|\frac{\cos(1000\pi y)+\cos(2023 y)}{2}\right|$$

$f(x)=g(\sin x)$ PER. E CONTINUA.

$f(x_0)$