Lezione 24: Differenziabilità (II)

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DERIVATE DI $f(x)$ ELEMENTARI

1) $f(x)=c \Rightarrow f'(x)=0$

$\boxed{x_0}$

$$f'(x_0)=\lim_{x\to x_0}\frac{\overset{c\,-\,c}{f(x)-f(x_0)}}{x-x_0}=\lim_{x\to x_0}\frac{0}{x-x_0}=\lim_{x\to x_0}0=0$$

2) $f(x)=x \Rightarrow f'(x)=1$

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{\cancel{x-x_0}}{\cancel{x-x_0}}=\lim_{x\to x_0}1=1$$

3) $f(x)=x^n \Rightarrow \boxed{f'(x)=n\,x^{n-1}}$

$\boxed{n=1}$ Si

$\boxed{n=k \Rightarrow n=k+1}$ $f(x)=x^{k+1}$

$$f'(x)=\left(x^{k+1}\right)'=\left(\boxed{x^k}\,x\right)'=k\cdot x^{k-1}\cdot x+x^k\cdot 1=$$

$$=\boxed{(k+1)x^k}$$

4) $f(x)=e^x \Rightarrow f'(x)=e^x$

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{e^x-e^{x_0}}{x-x_0}=$$

$$\lim_{x\to x_0}\underbrace{\boxed{e^{x_0}}}_{\boxed{e^{x_0}}}\overbrace{\frac{e^{x-x_0}-1}{x-x_0}}^{\to 1}=e^{x_0}\cdot 1=e^{x_0}$$

5) $f(x)=\ln x \Rightarrow f'(x)=\frac{1}{x}$ $\quad (x>0)$

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{\ln x-\ln x_0}{x-x_0}=$$

$$=\lim_{x\to x_0}\frac{\ln(x-x_0+x_0)-\ln x_0}{x-x_0}=$$

$x-x_0=h$

$$\downarrow$$

$$=\lim_{h\to 0}\frac{\ln(x_0+h)-\ln x_0}{h}=$$

$$=\lim_{h\to 0}\underbrace{\boxed{\frac{\ln\left(1+\frac{h}{x_0}\right)}{\frac{h}{x_0}}}}_{\to 1}\cdot\underbrace{\boxed{\frac{1}{x_0}}}_{\to \frac{1}{x_0}}=1\cdot\frac{1}{x_0}=\frac{1}{x_0}$$

6) $f(x)=\sin x \Rightarrow f'(x)=\cos x$

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{\sin(x_0+h)-\sin x_0}{h}=$$

$$=\lim_{h\to 0}\frac{\overbrace{\sin x_0\cdot\cos h}+\cos x_0\cdot\sin h\,\overbrace{-\sin x_0}}{h}=$$

$$=\lim_{h\to 0}\left(\underbrace{\sin x_0}_{\to\sin x_0}\cdot\underbrace{\boxed{\frac{\cos h-1}{h^2}}}_{\to -\frac{1}{2}}\cdot\underbrace{h}_{\to 0}+\underbrace{\cos x_0}_{\to\cos x_0}\cdot\underbrace{\frac{\sin h}{h}}_{\to 1}\right)=\cos x_0$$

7) $f(x)=\cos x \Rightarrow f'(x)=-\sin x$

$$f'(x_0)=\lim_{h\to 0}\frac{\cos(x_0+h)-\cos x_0}{h}=$$

$$= \lim_{h\to 0} \frac{\cos x_0 \cdot \cos h - \sin x_0 \cdot \sin h - \cos x_0}{h} =$$

$$= \lim_{h\to 0} \left(\underbrace{\cos x_0}_{\downarrow\ \cos x_0} \cdot \underbrace{\frac{\cos h - 1}{h^2}}_{\downarrow\ -\frac{1}{2}} \cdot \underbrace{h}_{\downarrow\ 0} - \underbrace{\sin x_0}_{\downarrow\ -\sin x_0} \underbrace{\frac{\sin h}{h}}_{\downarrow\ 1} \right) = -\sin x_0$$

8) $f(x) = a^x \Rightarrow f'(x) = a^x \ln a$

$$(a^x)' = \left(e^{x \ln a}\right)' = e^{x \ln a} \cdot (x \ln a)' = a^x \cdot \ln a$$

$$\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x)$$

9) $f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{\ln a} \cdot \frac{1}{x}$

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' =$$

$$= \frac{1}{\ln a} \cdot \frac{1}{x}$$

10) $f(x) = x^\alpha \Rightarrow f'(x) = \alpha x^{\alpha - 1}$ $\quad$ $x > 0$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{(x_0+h)^\alpha - x_0^\alpha}{h} =$$

$$= \lim_{h \to 0} \underbrace{\boxed{x_0^{\alpha-1}}}_{\downarrow\, x_0^{\alpha-1}} \cdot \underbrace{\boxed{\frac{\left(1+\frac{h}{x_0}\right)^\alpha - 1}{\frac{h}{x_0}}}}_{\downarrow\, \alpha} = \alpha x^{\alpha-1}$$

$$\boxed{\alpha = \frac{1}{2} \qquad x^\alpha = \sqrt{x} \qquad \alpha x^{\alpha-1} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}}$$

(11) $f(x) = \tan x \Rightarrow f'(x) = \underline{1 + \tan^2 x}$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \boxed{1 + \tan^2 x} \;\;\text{or}\;\; \boxed{\frac{1}{\cos^2 x}}$$

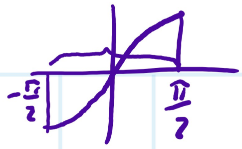
(12) $f(y) = \arctan y \Rightarrow f'(y) = \dfrac{1}{1+y^2}$

$$D(\arctan(y))\Big|_{y=y_0} = \frac{1}{D(\tan x)\Big|_{x=x_0}} =$$

$$\uparrow \; \tan x_0 = y_0$$

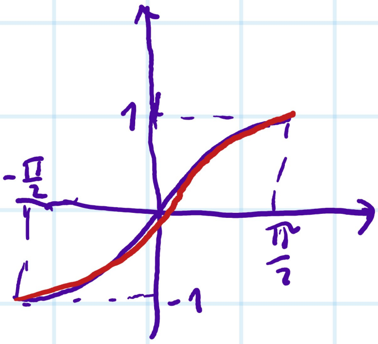
$$= \frac{1}{1 + \tan^2 x_0} = \frac{1}{1 + y_0^2}$$

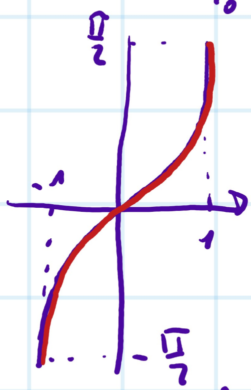
13) $f(y) = \arcsin y \Rightarrow f'(y) = \frac{1}{\sqrt{1-y^2}}$



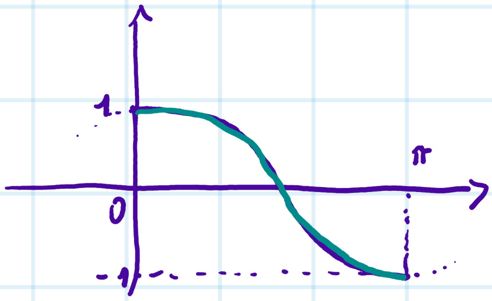
$$D(\arcsin y)\Big|_{y=y_0} = \frac{1}{D(\sin x)\Big|_{x=x_0}} = \frac{1}{\cos x_0} = \frac{1}{\sqrt{1-\sin^2 x_0}} = \frac{1}{\sqrt{1-y_0^2}}$$

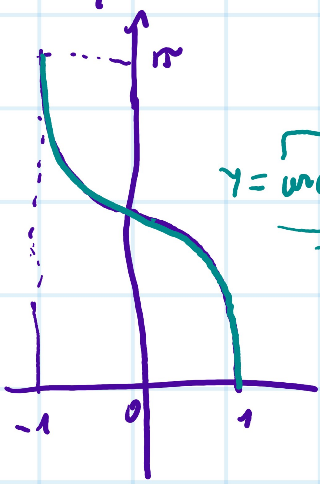
$y_0 = \sin x_0$





$y = \arcsin x$





$y = \arccos x = \frac{\pi}{2} - \arcsin x$

$$(\arccos x)' = \left(\frac{\pi}{2} - \arcsin x\right)' = 0 - \frac{1}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}$$

Es. per casa

1) Trovare inversa di $\tan x$ presa tra $-\frac{5}{2}\pi$ e $-\frac{3}{2}\pi$

2) . . . . $\sin x$ presa tra $\frac{9}{2}\pi$ e 5π

Es. 1

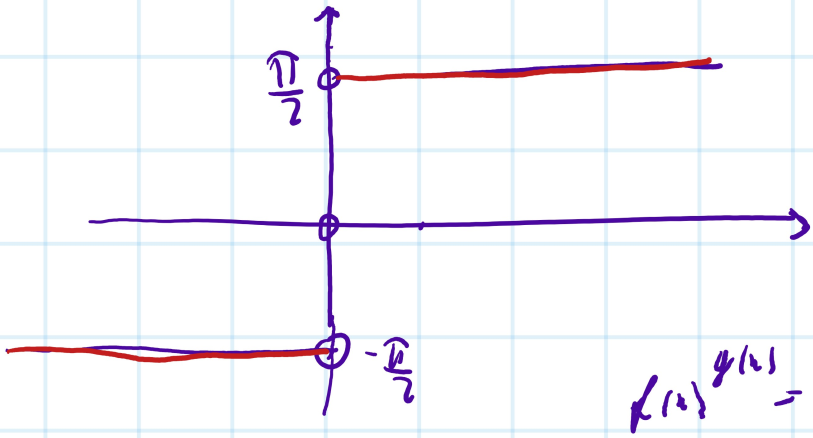
$$\left(\boxed{\arctan x + \arctan \frac{1}{x}}\right)' =$$

$$= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot(-1)\cdot\frac{1}{x^2} =$$

$$= \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0$$

$\alpha = -1$

$$(x^\alpha)' = \alpha x^{\alpha-1} = -1\cdot x^{-2} = -\frac{1}{x^2}$$



$$f(x)^{g(x)} = e^{\underline{g(x)\cdot \ln f(x)}}$$

Es. 2

$$(x^x)' = (e^{x\ln x})' =$$

$$= \underline{e^{x\ln x}}\cdot(x\ln x)' =$$

$$= x^x\cdot\left(1\cdot\ln x + x\cdot\frac{1}{x}\right) =$$

$$= x^x\cdot(\ln x + 1)$$

Per casa $\left(x^{x^x}\right)' = ??$ $\qquad x^{x^{x^{\cdot^{\cdot^{x}}}}}$ (n)

T. ST. SU DER.

1 T. FERMAT

DATI $A \subset \mathbb{R}$, $x_0 \in \mathring{A}$, $f: A \to \mathbb{R}$ DERIV. IN x_0 E CON x_0 CHE È PER f DI MAX. O DI MIN. ALLORA $f'(x_0)=0$

DIM (NEL CASO x_0 È PT. DI MAX.)

$f'_+(x_0) = f'_-(x_0)$ perché f è derivabile in x_0

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = \text{È} \leq 0$$

SEMPRE ≤ 0

f è derivabile $\to$ ||

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = \text{È} \geq 0$$

≤ 0, < 0, ≥ 0

$\Rightarrow \boxed{=0}$

T. ROLLE

DATA $f:[a,b] \to \mathbb{R}$ t.c.
1) CONTINUA SU $[a,b]$
2) DERIVABILE SU (a,b)
3) $f(a)=f(b)$

ALLORA $\exists c \in (a,b)$ t.c. $f'(c)=0$

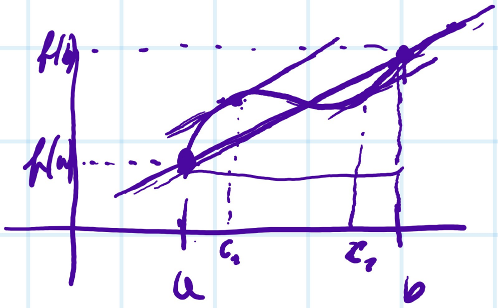
DIM $\exists^{no}$ x_1 = P.T. DI MAX x_2 = P.T. DI MIN. (PER WEIERSTRASS)

2 CASI $\to$ ENTRAMBI SUL BORDO $\Rightarrow f(x_1)=f(x_2) \Rightarrow$ MAX = MIN $\Rightarrow f$ cost. $\Rightarrow f'=0$ sempre

$\to$ ALMENO UNO È INTERNO $\Rightarrow$ IN TALE PUNTO $f'=0$ (PER FERMAT)

T. (LAGRANGE) DATA $f:[a,b]\to\mathbb{R}$ cont. in $[a,b]$ E DER. su (a,b)

ALLORA $\exists\, c \in (a,b)$ t.c. $f'(c)=\dfrac{f(b)-f(a)}{b-a}$



$\boxed{\text{DIM}}$

$$F(x)=f(x)-\underbrace{\left(f(a)+\boxed{\frac{f(b)-f(a)}{b-a}}(x-a)\right)}$$

A $F(x)$ POSSO APPLICARE ROLLE PERCHÉ È CONT. E DER. DOVE SERVE

E $\boxed{F(b)=F(a)}$

$$F(b)=f(b)-\left(f(a)+\frac{f(b)-f(a)}{\cancel{b-a}}\cdot \cancel{b-a}\right)=0$$

$$F(a)=f(a)-\left(f(a)+\frac{f(b)-f(a)}{b-a}(a-a)\right)=0$$

APPL. ROLLE SI HA CHE $\exists\, c\in(a,b)$ t.c.

$$F'(c)=0$$

$$F'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$$

$$\Downarrow$$

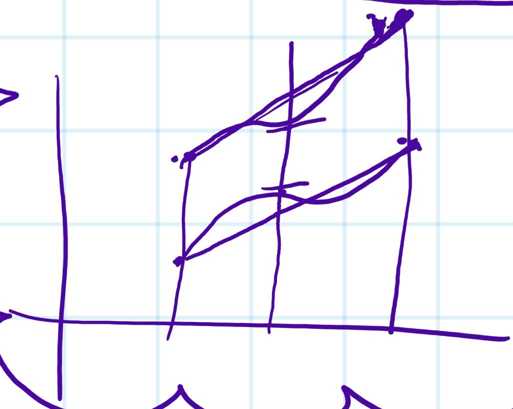
$$f'(c)-\frac{f(b)-f(a)}{b-a}=0 \Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$$

T. CAUCHY DATE $f, g : [a,b] \to \mathbb{R}$ CONTINUE SU $[a,b]$ E DERIVABILI SU (a,b)

ALLORA $\exists c \in (a,b)$ t.c.

$$f'(c)\cdot(g(b)-g(a)) = g'(c)\cdot(f(b)-f(a))$$

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}}}$$



DIM $H(x) = f(x)\cdot(g(b)-g(a)) - g(x)\cdot(f(b)-f(a))$

$H(x)$ REGOLARE QUANTO SERVE

$H(a) \overset{?}{=} H(b)$

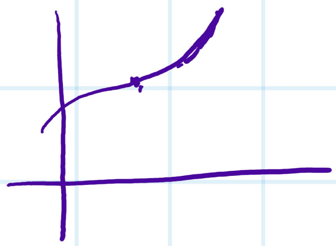
$$H(a) = f(a)\cdot(g(b)-\cancel{g(a)}) - g(a)(f(b)-\cancel{f(a)}) = f(a)g(b) - g(a)f(b)$$

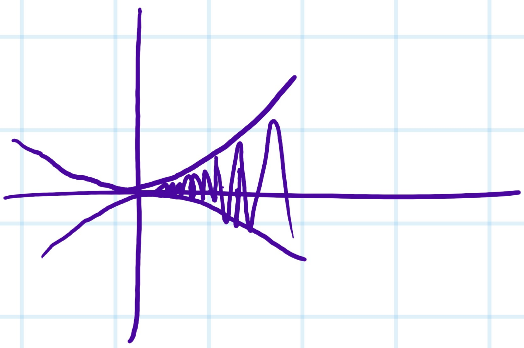
$$H(b) = f(b)(\cancel{g(b)} - g(a)) - g(b)(\cancel{f(b)} - f(a)) = -f(b)g(a) + g(b)f(a)$$

$\exists c \in (a,b)$ t.c. $H'(c) = 0$

$$\underline{f'(c)(g(b)-g(a))} - \underline{g'(c)(f(b)-f(a))} = 0$$

1ª APP. DI T.-L.





$$f(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin\frac{1}{x} & x \neq 0 \end{cases} \qquad f'(0) = 0$$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{x} = 0$$

$$f'(x) = \begin{cases} 0 & x = 0 \\ \underbrace{2x\sin\frac{1}{x}}_{\to 0} - \cos\frac{1}{x} & x \neq 0 \end{cases}$$

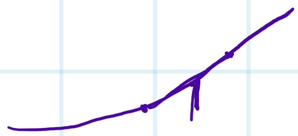
$$\left(x^2 \cdot \sin\frac{1}{x}\right)' = 2x\sin\frac{1}{x} + \cancel{x^2} \cdot \cos\frac{1}{x} \cdot \left(-\frac{1}{\cancel{x^2}}\right) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

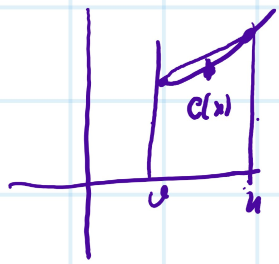
T. DATA $f:[a,b)\to\mathbb{R}$, DER. SU (a,b) E TALE CHE

$\lim_{x\to a^+} f'(x) = \lambda \in \mathbb{R}$ ALLORA SI PUÒ ESTENDERE

CON CONTINUITÀ IN a IN MODO CHE $f'_+(a) = \lambda$.

DIM SU $(a, a+\delta)$ DERIVATA È LIMITATA





$$f'(c_x) = \frac{f(x)-f(a)}{x-a}$$

$a < c(x) < x$

$$\lim_{x\to a^+} \boxed{\frac{f(x)-f(a)}{x-a}} = \lim_{x\to a^+} f'(\overbrace{c(x)}^{c}) = \lim_{c\to a^+} f'(c) = \boxed{\lambda}$$