Lezione 28: Polinomio di Taylor (II)

INDICE

1) SVILUPPI DI TAYLOR DI ALCUNE FUNZIONI ELEMENTARI :

$$\left(e^x,\ \sin x,\ \cos x,\ \frac{1}{1-x},\ \ln(1+x),\ \arctan x,\ \underline{(1+x)^\alpha}\right)$$

2) UTILIZZO NEL CALCOLO DI LIMITI

$$e^x = 1 + x + o(x)$$

$$e^x = \boxed{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!}} + o(x^n) \qquad (?)$$

$$\frac{x^{n+1}}{(n+1)!} + o(x^{n+1})$$

$$T(x) = \underbrace{a_0 + a_1 x + \dots + a_n x^n}$$

$$a_k = \frac{f^{(k)}(0)}{k!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!}$$

$$f(x) = e^x$$

$$\vdots$$

$$f^{(k)}(x) = e^x$$

$$a_k = \frac{e^0}{k!} = \frac{1}{k!}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!} + O(x^{n+1})$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \boxed{o(x^{2k+2})} \qquad \boxed{O(x^{2k+3})}$$

$$\boxed{\sin x = x + o(x)}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + O(x^{2k+2})$$

$$\boxed{\cos x = 1 - \frac{x^2}{2} + o(x^2)}$$

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + o(x^n) \quad O(x^{n+1})}$$

$$\frac{1}{1-x} - (1 + x + x^2 + \cdots + x^n) =$$

$$= \frac{1}{1-x} - \frac{1-x^{n+1}}{1-x} =$$

$$= \frac{x^{n+1}}{1-x} = O(x^{n+1})$$

$$1 + x + x^2 + \cdots + x^n = \boxed{\frac{1-x^{n+1}}{1-x}} \; (?)$$

$$\underbrace{(1 + x + x^2 + \cdots + x^n)(1-x)}_{} \overset{?}{=} \underbrace{1 - x^{n+1}}_{}$$

$$\boxed{1} + \cancel{x} + \cdots + \cancel{x^n} - \cancel{x} - \cancel{x^2} - x^3 - \cdots - \cancel{x^n} - x^{n+1}$$

$$= \boxed{1 - x^{n+1}}$$

$$\boxed{\frac{\frac{\cancel{x^{n+1}}}{1-x}}{\cancel{x^{n+1}}}} = \frac{1}{1-x} \to 1$$

$\boxed{\text{per } x \to \pi}$

$$\underbrace{e^{\sin x}} - \underbrace{\left(1 + \sin x + \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{6}\right)} = O\left((\sin x)^4\right)$$

$$o\left((\sin x)^3\right) = o(x^3)$$

$$\lim_{x \to \pi} \frac{e^{\sin x} - \left(1 + \sin x + \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{6}\right)}{(\sin x)^3} \underset{\uparrow}{=} \lim_{y \to 0} \frac{e^y - \left(1 + y + \frac{y^2}{2} + \frac{y^3}{6}\right)}{y^3} =$$

$y = \sin x$

$\boxed{\text{PER } x\to x_0}$

$$\underbrace{e^{f(x)}} = \underbrace{1+f(x)+\frac{(f(x))^2}{2}+\dots+\frac{(f(x))^n}{n!}} + \boxed{o\left((f(x))^n\right)}$$

SE $f(x)\to 0$ PER $x\to x_0$

$$\frac{(f(x))^{n+1}}{(n+1)!} + o\left((f(x))^{n+1}\right)$$

$\sin(f(x)) = \dots$

$\cos(f(x)) \dots$

$$\frac{1}{1-(f(x))} \dots$$

$$\underline{\frac{1}{1+x}} = \frac{1}{1-(-x)} = 1+(-x)+(-x)^2+\dots+(-x)^n+o(x^{n+1}) =$$

$f(x)$

$$= \underline{1-x+x^2-x^3+\dots+(-1)^n x^n} + o(x^{n+1})$$

$\boxed{F'(x)=f(x)}$

$$f(x) = \boxed{a_0+a_1x+\dots+a_nx^n} + o(x^{n+1}) \qquad \boxed{x\to 0}$$

$$F(x) = \underline{A_0+A_1x+\boxed{A_kx^k}+A_nx^n+\boxed{A_{n+1}x^{n+1}}} + o(x^{n+2})$$

$$(f(x))^{k-1}\Big|_{x=0} = (F(x))^k\Big|_{x=0} = \left(\quad \square \quad\right)^k\Big|_{x=0}$$

$$\| $$

$$\left(\quad \square \quad\right)^{k-1}\Big|_{x=0}$$

$$\underbrace{\frac{1}{1+x}} = \underbrace{1-x+x^2-x^3+\dots+(-1)^n x^n} + o(x^{n+1})$$

$$\underbrace{\ln(1+x)} = \_\_\_ + o(x^{n+2})$$

$$P'(x) = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n$$

$$P(0) = 0$$

$$P(x) = \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^n} \frac{x^{n+1}}{n+1}$$

$$\frac{F(x) - T(x)}{(x-x_0)^{n+1}} \to 0$$

$$\frac{f(x) - (T' \dots)}{n+1\,(x-x_0)^n} \to 0$$

$$(\arctan x)' = \boxed{\frac{1}{1+x^2}}$$

$$\frac{1}{1+x^2} = \frac{1}{1-\underbrace{(-x^2)}_{f(x)}} = 1 + (-x^2) + (-x^2)^2 + \dots + (-x^2)^n + O(x^{2n+2})$$

$$= 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + O(x^{2n+1})$$

$$\frac{1}{1-x}$$

$$(P(x))' = \square$$

$$0 = \arctan(0) = P(0)$$

$$P(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + O(x^{2n+3})$$

$$\binom{\boxed{\alpha}}{k}$$

$$\binom{n}{k} = \boxed{\frac{n!}{k!(n-k)!}} = \frac{n(n-1)\cdots(n-k+1)\overline{(n-k)\cdots 3\cdot 2\cdot 1}}{k!\,(n-k)!} = \boxed{\frac{n(n-1)\cdots(n-k+1)}{k!}}$$

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n$$

$$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \cdots + \boxed{\binom{\alpha}{k}}x^k + \cdots \binom{\alpha}{n}x^n + O(x^{n+1})$$

$$a_k x^k$$

$$\boxed{\frac{f^{(k)}(0)}{k!}} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} = \boxed{\binom{\alpha}{k}}$$

$$f^{(k)}(x) = \left((1+x)^\alpha\right)^{(k)} = \left(\alpha(1+x)^{\alpha-1}\right)^{(k-1)} = \cdots =$$

$$= \alpha(\alpha-1)\cdots(\alpha-k+1)\cdot(1+x)^{\alpha-k}$$

$$f^{(k)}(0) = \alpha(\alpha-1)\cdots(\alpha-k+1)$$

$$59 \quad \lim_{x\to 0} \frac{\ln\left(1+\frac{x}{2}\right) - \sqrt{1+x} + 1}{2(\tan x - \sin x)} =$$

$$= \lim_{x\to 0} \frac{\cancel{\frac{x}{2}} - \cancel{\frac{x^2}{8}} + \frac{x^3}{24} + O(x^4) - \left(\cancel{1} + \cancel{\frac{x}{2}} - \cancel{\frac{x^2}{8}} + \frac{x^3}{16} + O(x^4)\right) \cancel{+1}}{x^3}$$

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + O(y^4)$$

$$\ln\left(1+\frac{x}{2}\right) = \frac{x}{2} - \frac{\left(\frac{x}{2}\right)^2}{2} + \frac{\left(\frac{x}{2}\right)^3}{3} + O(x^4) =$$

$$\boxed{\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)} \qquad = \boxed{\frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} + O(x^4)}$$

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1}x + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3 + O(x^4) =$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{2!}\cdot x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 +$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{6}x^3 + O(x^4) =$$

$$= \boxed{1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}x^3 + O(x^4)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{48}x^3 + O(x^4)}{x^3} = \lim_{x\to 0} \frac{-\frac{1}{48}\cancel{x^3}}{\cancel{x^3}} = -\frac{1}{48}$$