Analisi Matematica 1 - 11 Dicembre 2023 - docente: Callegari

Lezione 30: Polinomio di Taylor (IV)

INDICE

$\boxed{\text{T.}}$ (TAYLOR CON RESTO DI LAGRANGE)

DATI $\underbrace{}_{(n \in \mathbb{N})}$, $f \in C^{n+1}((a,b))$ $x_0 \in (a,b)$, ALLORA

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \quad (\text{Con } c \text{ compreso tra } x_0 \text{ e } x)$$

POL. DI TAYLOR DI ORDINE n DI f CENTRATO IN x_0

PER $n=0$ SI RITROVA IL T. DI LAGRANGE

$$f(x) - f(x_0) = f'(c)\boxed{(x-x_0)} \qquad c \in (x_0, x)$$

$\boxed{\text{DIM}}$ $F(x)$, $G(x)$ $\qquad \dfrac{F(x)-F(x_0)}{G(x)-G(x_0)} = \dfrac{F'(c)}{G'(c)} \qquad c \in (x_0, x)$

$F(x) = f(x) - T_n(x) \qquad F(x_0) = 0$

$G(x) = (x-x_0)^{n+1} \qquad G(x_0) = 0$

$$\boxed{\dfrac{f(x)-T_n(x)}{(x-x_0)^{n+1}}} = \ldots\ldots = \dfrac{f^{(n+1)}(c)}{(n+1)!}$$

$\|$

$$\dfrac{\overbrace{[f(x)-T_n(x)] - (f(x_0)-T_n(x_0))}^{F(x)-F(x_0)}}{\underbrace{(x-x_0)^{n+1} - (x_0-x_0)^{n+1}}_{G(x)-G(x_0)}} = \dfrac{F'(x_1)}{G'(x_1)} = \dfrac{\overbrace{f'(x_1) - \overbrace{T_n'(x_1)}^{T_{n-1,f'}(x)}}}{(n+1)(x_1-x_0)^n} = (*)$$

con x_1 tra x_0 e x

$$\dfrac{h(x) - T_{n-1}(x)}{(x-x_0)^n} = \dfrac{h^{(n)}(c)}{n!}$$

$$T'_{n,f}(x) = \left(\sum_{k=0}^{n} \dfrac{f^{(k)}(x_0)}{k!}\cdot(x-x_0)^k\right)' = \sum_{k=1}^{n} \dfrac{f^{(k)}(x_0)}{\cancel{k!}_{(k-1)!}} \cdot \cancel{k}(x-x_0)^{k-1}$$

$k-1=m$

$$= \sum_{m=0}^{n-1} \frac{(f')^{(m)}(x_0)}{m!} \cdot (x-x_0)^m = T_{n-1,f'}(x)$$

$$(*) = \frac{1}{(n+1)} \cdot \frac{(f')^{(n)}(c)}{n!} =$$

c COMPRESO TRA x_1 E x_0

e quindi compreso tra x e x_0

$$= \boxed{\frac{f^{(n+1)}(c)}{(n+1)!}}$$

ES.

$$e^x - T_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$$

$\boxed{c \text{ tra } x \text{ e } x_0}$

$$e^x - \sum_{k=0}^{n} \frac{x^k}{k!} = \frac{e^c}{(n+1)!} x^{n+1}$$

$c \in (0,1)$

$$e - \sum_{k=0}^{n} \frac{1}{k!} = \frac{e^c}{(n+1)!}$$

$$0 \leq \left| e - \underbrace{\sum_{k=0}^{n} \frac{1}{k!}}_{S_n} \right| \leq \frac{e}{(n+1)!}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 \qquad\qquad\qquad 0$$

$$\sum_{k=0}^{+\infty} a_k = \lim_{n\to+\infty} \left(\sum_{k=0}^{n} a_k\right) = \lim_{n\to+\infty} S_n$$

$$S_n - e \to 0$$

$$S_n \to e$$

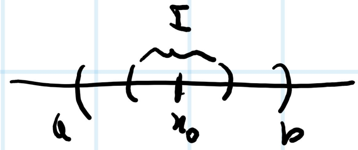
c tra $\underset{\uparrow}{\overset{x_0}{}}0$ e $\underset{\uparrow}{\overset{x}{}}\alpha$

$$\underbrace{0}_{\downarrow\, 0} \le \underbrace{\left| e^{\alpha} - \sum_{k=0}^{n} \frac{\alpha^k}{k!} \right|}_{\downarrow\, 0} \le \underbrace{\frac{e^c}{(n+1)!}\,\alpha^{n+1}}_{\downarrow\, 0}$$

$$\boxed{e^{\alpha}} = \lim_{n \to +\infty} \sum_{k=0}^{n} \frac{\alpha^k}{k!} = \sum_{k=0}^{+\infty} \frac{\alpha^k}{k!}$$

$$f(x) = e^x$$

$$F(x) = x \mapsto \boxed{\sum_{k=0}^{+\infty} \frac{x^k}{k!}}$$



$\boxed{\text{DEF.}}$ DATA $f \in C^{\infty}(a,b)$ SIA $x_0 \in (a,b)$ E SI SUPPONGA CHE $\forall x \in I \overset{\text{APERTO}}{\subset} (a,b)$ t.c. $x_0 \in I$ SI ABBIA

$$\lim_{n \to +\infty} \left(\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \right) = f(x)$$

ALLORA DIREMO CHE $f(x)$ È ANALITICA SU I

ESEMPIO CATTIVO

$$\boxed{f(x)=\begin{cases} 0 & \text{SE } x=0 \\ e^{-\frac{1}{x^2}} & \text{SE } x\neq 0 \end{cases}}$$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x}=$$

$$=\lim_{x\to 0}\frac{\frac{1}{x}}{e^{\left(\frac{1}{|x|}\right)^2}}=\lim_{y\to 0^+}\frac{y}{e^{y^2}}=0$$

$$f'(x)=\begin{cases} 0 & x=0 \\ \boxed{2\frac{1}{x^3}}e^{-\frac{1}{x^2}} & x\neq 0 \end{cases}$$

$$\boxed{e^{-\frac{1}{x^2}}\cdot 2\cdot\frac{1}{x^3}}$$

$$(-x^{-2})'=-(-2)$$

$$p(y)=2y^3$$

$$f''(x)=\begin{cases} 0 & x=0 \\ \underbrace{e^{-\frac{1}{x^2}}\cdot\underbrace{\left(-\frac{6}{x^4}+\frac{4}{x^6}\right)}} & x\neq 0 \end{cases}$$

$$\lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0}=$$

$$=\lim_{x\to 0}\frac{2\frac{1}{x^3}e^{-\frac{1}{x^2}}}{x}=$$

$$=\lim_{x\to 0}2\cdot\frac{\frac{1}{x^4}}{e^{+\frac{1}{x^2}}}=$$

$$=0$$

$$\boxed{p(y)=-6y^4+4y^6}$$

$$\left(2\frac{1}{x^3}\cdot e^{-\frac{1}{x^2}}\right)'=$$

$$=-6\cdot\frac{1}{x^4}\cdot e^{-\frac{1}{x^2}}+\frac{2}{x^3}\cdot e^{-\frac{1}{x^2}}\cdot\frac{2}{x^3}=$$

$$=e^{-\frac{1}{x^2}}\left(-\frac{6}{x^4}+\frac{4}{x^6}\right)$$

$$f^{(n)}(x)=\begin{cases} 0 & x=0 \\ \underbrace{e^{-\frac{1}{x^2}}\cdot p\left(\frac{1}{x}\right)} & x\neq 0 \end{cases}$$

$$f^{(n+1)}(x) = \begin{cases} 0 \\ \boxed{} \end{cases}$$

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} \cdot p\left(\frac{1}{x}\right)}{x} =$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}\cdot p\left(\frac{1}{x}\right)}{e^{\left(\frac{1}{x}\right)^2}} \overset{y=\frac{1}{x}}{=} \lim_{y\to\pm\infty} \frac{\overline{y\cdot p(y)}}{e^{y^2}} = 0$$

$$\left(p\left(\frac{1}{x}\right)\cdot e^{-\frac{1}{x^2}}\right)' = p'\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)\cdot e^{-\frac{1}{x^2}} + p\left(\frac{1}{x}\right)\cdot$$

$$\cdot e^{-\frac{1}{x^2}}\cdot\left(\frac{2}{x^3}\right) =$$

$$\boxed{q(y) = \left(-y^2 p'(y) + p(y)\cdot 2y^3\right)} = e^{-\frac{1}{x^2}}\left(\underline{\overline{-\frac{1}{x^2}\cdot p'\left(\frac{1}{x}\right) + p\left(\frac{1}{x}\right)\cdot\frac{2}{x^3}}}\right) =$$

$$= e^{-\frac{1}{x^2}} \cdot \underline{q\left(\frac{1}{x}\right)}$$

$$\boxed{f^{(n)}(0) = 0 \qquad \forall n \in \mathbb{N}}$$

$$T_n(x) = \overset{\overset{0}{\downarrow}}{f(0)} + \overset{\overset{0}{\downarrow}}{f'(0)}\,x + \frac{\overset{\overset{0}{\downarrow}}{f''(0)}}{2!}\cdot x^2 + \dots + \frac{\overset{\overset{0}{\downarrow}}{f^{(n)}(0)}}{n!}\,x^n =$$

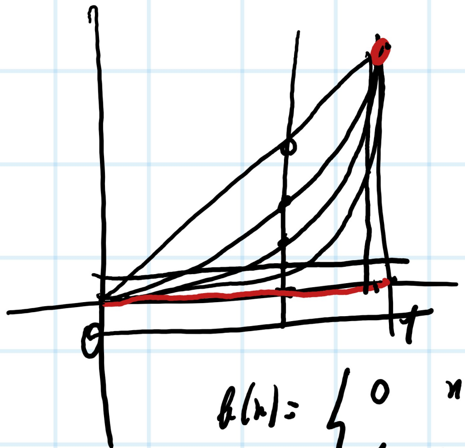
$$= 0$$

$$F(x) = \sum_{k=0}^{+\infty} \frac{\overset{\overset{0}{\downarrow}}{f^{(k)}(0)}}{k!}\cdot x^k = \sum_{k=0}^{+\infty} 0\cdot x^k = \sum_{k=0}^{+\infty} 0 = 0$$

$\boxed{f(x) = e^x}$

$$\boxed{T_n(x)} \xrightarrow{n\to+\infty} f(x)$$

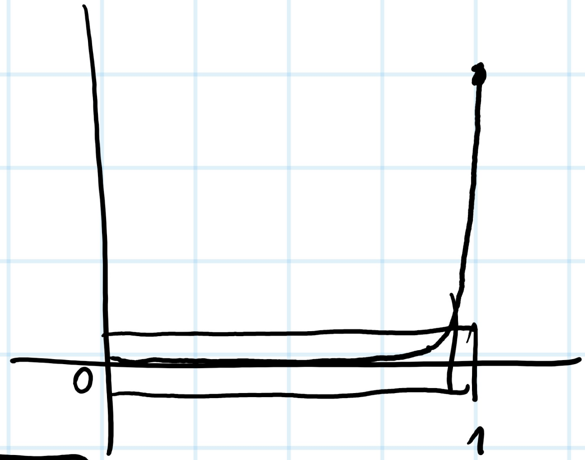
$$\to \lim_{n\to+\infty} \boxed{\sum_{k=0}^{n} \frac{x^k}{k!}} = e^x$$



$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

x

$$f_n(x) = x^n$$

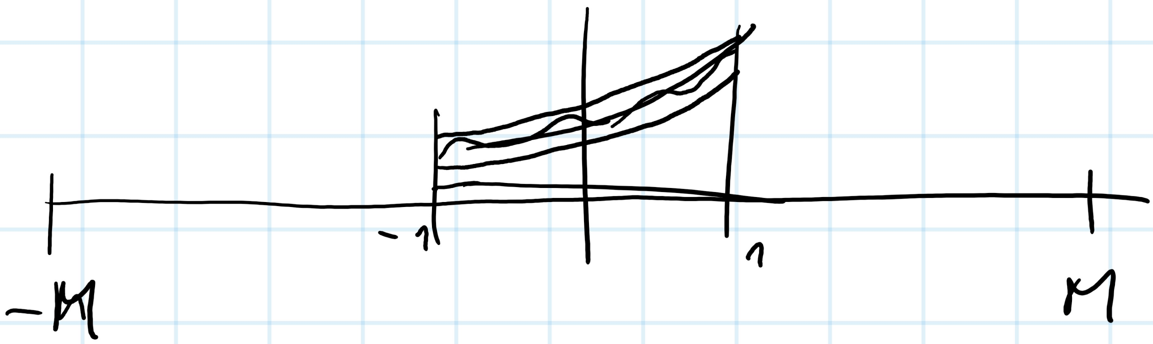


$x \in [-1,1]$

$\boxed{\forall\, x \in [-1,1]}$ $\boxed{c \text{ tra } 0 \text{ e } x}$

$$\underbrace{\left|e^x - T_n(x)\right|} \leq \frac{e^c}{(n+1)!}\cdot|x|^{n+1} \leq \boxed{\frac{e}{(n+1)!}} < \varepsilon$$

$$\sup_{x\in[-1,1]} \boxed{} \leq \frac{e}{(n+1)!}$$



$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots =$$

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} = 2$$

$$= 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots\right)$$

$$S = 1 + \frac{1}{2} S$$

$$\frac{1}{2} S = 1 \qquad S = 2$$

$$S = 1 + 2 + 4 + \cdots + 2^n + \cdots =$$

$$\sum_{k=0}^{+\infty} 2^k = +\infty$$

$$S = f(S)$$

$$= 1 + 2\left(1 + 2 + \cdots + 2^{n-1} + \cdots\right)$$

$$S = 1 + 2S \qquad -S = 1 \qquad S = -1$$

$$S_0 = 1$$

$$S_1 = 1 + \frac{1}{2}$$

$$S_2 = 1 + \frac{1}{2} + \frac{1}{4}$$

$$\vdots$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$$

$$S_{n+1} = 1 + \overbrace{\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}} =$$

$$= 1 + \frac{1}{2} S_n$$

$$f(x) = 1 + \frac{1}{2} x$$

$$\begin{cases} S_{n+1} = f(S_n) \\ S_0 = 1 \end{cases}$$

$$g = 1 + 2x$$

$$A_0 = 1$$

$$A_n = 1 + 2 + 2^2 + \cdots + 2^n$$

$$A_{n+1} = 1 + 2 + 2^2 + \cdots + 2^n + 2^{n+1} =$$

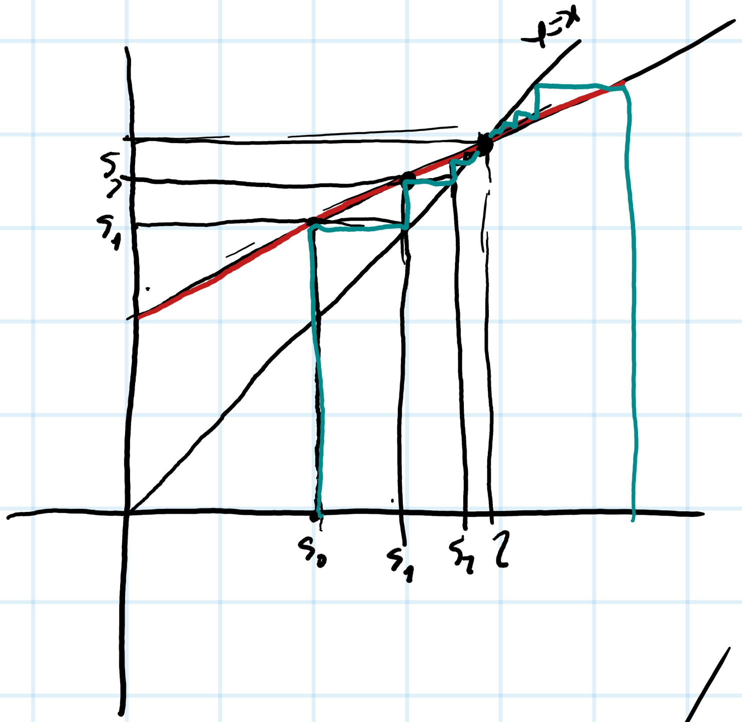
$$= 1 + 2(1 + 2 \cdots + 2^n) = 1 + 2A_n$$

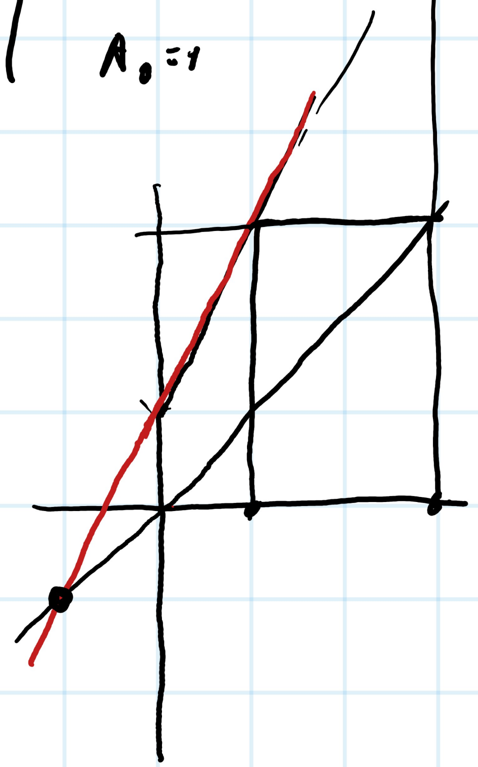
$$\begin{cases} A_{n+1} = g(A_n) \\ A_0 = 1 \end{cases}$$

$$\begin{cases} S_{n+1} = \overbrace{1+\frac{1}{2}S_n} = f(S_n) \\ S_0 = 1 \end{cases}$$

$g(x) = 1+2x$

$$\begin{cases} A_{n+1} = \overbrace{\underbrace{1+2A_n}} = g(A_n) \\ A_0 = 1 \end{cases}$$





$a_{n+1} = f(a_n)$