

Lezione 10: Esercizi sulle successioni

ANALISI MATEMATICA 1

(UNIVERSITÀ DI ROMA "TOR VERGATA"
CORSO DI LAUREA IN MATEMATICA A.A. 2024/2025
DOCENTE: CALLEGARI)

1

DOMANDE PER ESERCITARSI A CASA - 6 OTTOBRE 2024

I PARTE: USO DEI QUANTIFICATORI

A DIRE SE SONO VERE O FALSE LE SEGUENTI AFFERMAZIONI:

1 $\forall x \in \mathbb{R} \exists y \in \mathbb{R} \text{ tale che } y \leq x$

2 $\exists y \in \mathbb{R} \text{ tale che } \forall x \in \mathbb{R} y \leq x$

3 $\forall x \in \mathbb{N} \exists y \in \mathbb{N} \text{ tale che } y \leq x$

4 $\exists y \in \mathbb{N} \text{ tale che } \forall x \in \mathbb{N} y \leq x$

5 DEFINITIVAMENTE IN n , DEFINITIVAMENTE IN k $n \geq k$

6 DEFINITIVAMENTE IN k , DEFINITIVAMENTE IN n $n \geq k$

7 FREQUENTEMENTE IN n , FREQUENTEMENTE IN k k DIVIDE n

8 FREQUENTEMENTE IN k , FREQUENTEMENTE IN n k DIVIDE n

9 FREQUENTEMENTE IN n , DEFINITIVAMENTE IN k k DIVIDE n

10 DEFINITIVAMENTE IN k , FREQUENTEMENTE IN n k DIVIDE n

11 FREQUENTEMENTE IN k , DEFINITIVAMENTE IN n k DIVIDE n

12 DEFINITIVAMENTE IN n , FREQUENTEMENTE IN k k DIVIDE n

NOTAZIONI

DIRE CHE UN'AFFERMAZIONE CON UN PARAMETRO

 $n \in \mathbb{N}$ VALE "DEFINITIVAMENTE IN n "SIGNIFICA CHE " $\exists n_0 \in \mathbb{N} \text{ tale che } \forall n \geq n_0$ "L'AFFERMAZIONE VALE IN n .INVECE DIRE CHE VALE "FREQUENTEMENTE IN n "SIGNIFICA CHE " $\forall n_0 \in \mathbb{N} \exists n > n_0 \text{ tale che}$ "L'AFFERMAZIONE VALE IN n .B DATA UN'AFFERMAZIONE $P(n, k)$ DIPENDENTE DA 2 PARAMETRI $n, k \in \mathbb{N}$ DIRE, MOTIVANDO LA RISPOSTA,

QUALE DELLE 2 SEGUENTI PROPOSIZIONI È PIÙ FORTE:

1 $\forall k \in \mathbb{N} \text{ FREQUENTEMENTE IN } n \ P(n, k)$

2 FREQUENTEMENTE IN $n \ \forall k \in \mathbb{N} P(n, k)$

II PARTE: CONFRONTO DI INFINITI

A CONFRONTARE L'ORDINE DI INFINITO DEI SEGUENTI GRUPPI DI SUCCESSIONI:

1 $A_n = \log_2(n^2)$ $B_n = (\log_2(n))^2$ $C_n = \log_2(n)$ $D_n = \log_2(\sqrt{n})$ $E_n = \sqrt{\log_2(n)}$

2 $A_n = 10000^n$ $B_n = 10^{2^n}$ $C_n = 1000000^{\sqrt{n}}$ $D_n = 10^{n^2}$ $E_n = \sqrt{10^{n^2}}$

3 $A_n = (n+1000)!$ $B_n = n^{n-1000}$ $C_n = 2^{n^2}$ $D_n = n^{\sqrt{n}}$

4 $a_n = (5+3\sin n)^n$ $b_n = (7+2\sin n)^n$ $c_n = (6+3\cos n)^n$

→ **B** DIRE PER QUALI $\alpha \in (0,1)$ SI HA $n! = o((n^n)^\alpha)$.

C CALCOLARE I SEGUENTI LIMITI:

1 $\lim_{n \rightarrow +\infty} \frac{2^{\sqrt{n}} + \sqrt{(2n)^n} + \sin(n^{n^n})}{\sqrt{n^n + 2^{\frac{n}{2}}} - (\sqrt{n})^n}$

2 $\lim_{n \rightarrow +\infty} \left(\log_8(1+2^n) - \sqrt[n]{n+1} \right)$

3 $\lim_{n \rightarrow +\infty} \left(\sin(\sin n) \right)^{n^2} \cdot n^n$

D CONFRONTARE L'ORDINE DI INFINITO DEI SEGUENTI GRUPPI DI SUCCESSIONI:

→ 1 $A_n = \sqrt[n]{n!}$ $B_n = \sqrt{n}$ $C_n = \log_4(1+3^n)$ $D_n = (\log_2 n)^2$

2 $A_n = (n^n)^{n!}$ $B_n = (n^n)!$ $C_n = (n!)!$ $D_n = (n!)^{n!}$

B $\lim_{n \rightarrow +\infty} \frac{n!}{(h^n)^\alpha} = +\infty$
 $\leftarrow a_n$

T. (Rappresenta) DATA (a_n) SUPER. A VALORI IN $\mathbb{R}^+$ SI HA

$\exists k > 1$ t.p.
 DEF. in n $\frac{a_{n+1}}{a_n} > k$

1) $\frac{a_{n+1}}{a_n} \rightarrow l > 1 \Rightarrow a_n \rightarrow +\infty$

2) $\frac{a_{n+1}}{a_n} \rightarrow l < 1 \Rightarrow a_n \rightarrow 0$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{((n+1)^{n+1})^\alpha} \cdot \frac{(h^n)^\alpha}{n!} =$$

$\frac{(n+1)!}{(h+1)^{n+1} \cdot (h+1)^\alpha} \cdot \frac{(h^n)^\alpha}{n!}$



$$= \frac{n+1}{((h+1)^n)^\alpha \cdot (h+1)^\alpha} \cdot \frac{(h^n)^\alpha}{1} =$$

$$= \left(\left(\frac{h}{h+1} \right)^n \right)^\alpha \cdot (h+1)^{1-\alpha} = \left(\frac{h+1-1}{h+1} \right)^n \cdot (h+1)^{1-\alpha} =$$

$$= \left(1 - \frac{1}{h+1} \right)^n \cdot (h+1)^{1-\alpha} \rightarrow +\infty$$

$\downarrow \frac{1}{e}$

$$\left[\underbrace{\left(1 + \frac{h}{h+1} \right)^{n+1}}_{e^{-1}} \cdot \underbrace{\left(1 - \frac{1}{h+1} \right)^{-1}}_{\downarrow 1} \right] \rightarrow \frac{1}{e}$$

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

$$\left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n = \frac{1}{\left(\frac{n}{n-1}\right)^n} = \frac{1}{\underbrace{\left(1 + \frac{1}{n-1}\right)^n}_{b_{n-1}}} \rightarrow \frac{1}{e}$$

$$\left(1 + \frac{\lambda}{n}\right)^n \rightarrow e^\lambda \quad \lambda > 0$$

$$\boxed{\lambda < 0} \quad \left(1 + \frac{\lambda}{n}\right)^n = \left(\underbrace{\left(1 + \frac{1}{\frac{n}{\lambda}}\right)^{\frac{n}{\lambda}}}_e\right)^\lambda \rightarrow e^\lambda$$

$$\left(1 + \frac{1}{a_n}\right)^{a_n} \rightarrow e$$

$a_n \rightarrow +\infty$
 $a_n \rightarrow -\infty$

1 $A_n = \sqrt[n]{n!}$ $B_n = \sqrt{n}$ $C_n = \log_4(1+3^n)$ $D_n = (\log_2 n)^2$

$D_n = o(B_n)$ $B_n = o(A_n)$ A_n strom schneller als C_n

$$(\log_2 n)^2 = o(\sqrt{n})$$

$$\frac{(\log_2 n)^2}{n^{\frac{1}{2}}} = \left(\frac{\log_2 n}{n^{\frac{1}{4}}}\right)^2 \rightarrow 0$$

$B_n = o(A_n)$

$$0 \leq \frac{\sqrt{n}}{\sqrt[n]{n!}} \leq \frac{\sqrt{n}}{n^{\frac{3}{4}}} = \frac{1}{n^{\frac{1}{4}}} \rightarrow 0$$

$\sqrt[n]{a_n} > \sqrt[n]{b_n}$
 $0 < \alpha < 1$ $\sqrt[n]{n!} > \sqrt[n]{(n^\alpha)^{\frac{1}{2}}}$
 $n! \gg (n^\alpha)^{\frac{3}{4}} \Rightarrow n! \geq (n^\alpha)^{\frac{3}{4}}$
 $\sqrt[n]{n!} \geq \sqrt[n]{(n^\alpha)^{\frac{3}{4}}} = n^{\frac{\alpha}{4}}$

$$\begin{aligned}
 n^2 &\gg n \\
 \log_2(n^2) &\not\gg \log_2(n) \\
 2 \log_2(n) &\not\gg \log_2(n) \\
 \sqrt[n]{n!} &\gg \sqrt[n]{(n!)^{\frac{1}{2}}} = n^{\frac{1}{2}}
 \end{aligned}$$

"1" SI PUO' TOGLIERE

$$\frac{a_{n+1}}{a_n} \rightarrow l$$

$$\sqrt[n]{a_n} \rightarrow l$$

$$\log_4 3^n = (n) \log_4 3$$

$$\log_4 3^n = \frac{n}{\sqrt[n]{n!}}$$

$$\frac{n}{\sqrt[n]{n!}} = \sqrt[n]{\frac{n^n}{n!}}$$

$$\frac{\log_4(1+3^n)}{\log_4(3^n)} =$$

$$= \frac{\log_4 \left(3^n \left(1 + \frac{1}{3^n} \right) \right)}{\log_4 3^n} =$$

$$= \frac{\log_4(3^n) + \log_4 \left(1 + \frac{1}{3^n} \right)}{\log_4(3^n)} =$$

$$= \left(1 + \frac{\log_4 \left(1 + \frac{1}{3^n} \right)}{\log_4(3^n)} \right) \rightarrow 1$$

[T.]

DATI (a_n) A VALORI IN $\mathbb{R}^+$ SE $\frac{a_{n+1}}{a_n} \rightarrow l$ ALLORA ANCHE $\sqrt[n]{a_n} \rightarrow l$

INVERSE 1: $\sqrt[n]{\frac{h^n}{n!}} \rightarrow ?$

CASE 1

$$\frac{\frac{0_{n+1}}{a_n}}{\frac{(n+1)!}{h^n}} =$$

$$= \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{h!}{h^n} = \left(\frac{n+1}{h}\right)^h = \left(1 + \frac{1}{n}\right)^n \rightarrow e$$

⑦ $(0_n) (b_n)$ in $\mathbb{R}^+$

SE DEF. IN n $\frac{a_{n+1}}{a_n} > \lambda > 1$ ALLOW $a_n \rightarrow \infty$

IN

$\exists n_0 \in \mathbb{N}$ t.e. $\forall n \geq n_0$ $\frac{a_{n+1}}{a_n} > \lambda$

$$\frac{1}{a_{n_0}} \cdot a_{n_0+k} = \frac{a_{n_0+k}}{a_{n_0}} = \frac{a_{n_0+1}}{a_{n_0}} \cdot \frac{a_{n_0+2}}{a_{n_0+1}} \cdot \frac{a_{n_0+3}}{a_{n_0+2}} \cdots \frac{a_{n_0+k}}{a_{n_0+k-1}} > \lambda^k \rightarrow \infty$$

a_n $0 < \frac{a_{n+1}}{a_n} < \lambda < 1 \Rightarrow a_n \rightarrow 0$

$$b_n = \frac{1}{a_n}$$

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}} = \frac{a_n}{a_{n+1}} \rightarrow \frac{1}{\lambda} \Rightarrow b_n \rightarrow \infty$$

$$\left[\begin{array}{c} a_{n+1} \\ a_n \end{array} \right] \rightarrow l \Rightarrow \sqrt[n]{a_n} \rightarrow l \quad \left[\begin{array}{c} \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \\ \frac{a_{n+1}}{a_n} < l + \frac{\varepsilon}{2} \end{array} \right]$$

$$\forall \varepsilon > 0 \text{ DEFINITION } \sqrt[n]{a_n} < l + \varepsilon \quad ?$$

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \text{ t. } n > n_0 \Rightarrow \frac{a_{n+1}}{a_n} < l + \frac{\varepsilon}{2}$$

$$\sqrt[n]{a_n} = \sqrt[n]{a_{n_0} \cdot \frac{a_{n_0+1}}{a_{n_0}} \cdot \frac{a_{n_0+2}}{a_{n_0+1}} \cdots \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_n}{a_{n-1}}}$$

$$< \sqrt[n]{a_{n_0} \underbrace{\left(1 + \frac{\varepsilon}{2}\right) \cdots \left(1 + \frac{\varepsilon}{2}\right)}_{n - n_0}} = \sqrt[n]{a_{n_0} \cdot \left(1 + \frac{\varepsilon}{2}\right)^{n - n_0}} \quad A^{\frac{1}{n}} \rightarrow 0$$

$$= \sqrt[n]{a_{n_0}} \cdot \left(1 + \frac{\varepsilon}{2}\right)^{\frac{n - n_0}{n}} = \sqrt[n]{a_{n_0}} \cdot \left(1 + \frac{\varepsilon}{2}\right)^{1 - \frac{n_0}{n}} =$$

$$\underbrace{\left(a_{n_0}\right)^{\frac{1}{n}}}_{\downarrow 1} \cdot \underbrace{\left(1 + \frac{\varepsilon}{2}\right)}_{\downarrow 1 + \frac{\varepsilon}{2}} \cdot \underbrace{\frac{1}{\left(\left(1 + \frac{\varepsilon}{2}\right)^{n_0}\right)^{\frac{1}{n}}}}_{\text{constant factor}} \rightarrow l + \frac{\varepsilon}{2}$$

$$\left(\sqrt[n]{a_n} \leq b_n \right) \rightarrow \text{DEFINITION } \sqrt[n]{a_n} < l + \varepsilon$$

$b_n \downarrow l + \frac{\varepsilon}{2}$