Metodi Matematici - Ex. 3

Titolo nota

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ES. 23
LISTA 2

con $\langle \, , \, \rangle$

Dato $C([-1,1])$ e dato $H = \{ g(x) \mid g$ è polinomio di grado $\leq 2 \}$

e dato $f(x) = x^3$, trovare proiezione di f su H,

e trovare $d(f, H)$.

Prendo per H la base $\{1, x, x^2\} = B$

costruisco nuova base $\mathcal{B}$ che sia ortonormale.

1) $\| 1 \|_2^2 = \int_{-1}^{1} 1^2 \, dx = 2$

$$\frac{1}{\| 1 \|_2} = \frac{1}{\sqrt{2}}$$

$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}, \ldots\ldots \right\}$

2) x è $\perp$ $\frac{1}{\sqrt{2}}$ perché $\langle x, \frac{1}{\sqrt{2}} \rangle = \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}} \, dx = 0$

$$\| x \|_2^2 = \int_{-1}^{1} x^2 \, dx = \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3} \right) = \frac{2}{3}$$

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}, \frac{x}{\sqrt{\frac{2}{3}}}, \ldots\ldots \right\}$$

3) x^2 $\quad \Pi_{x^2}(x) = \overbrace{\left\langle x^2, \frac{1}{\sqrt{2}} \right\rangle \cdot \frac{1}{\sqrt{2}}} + \left\langle x^2, \frac{x}{\sqrt{\frac{2}{3}}} \right\rangle \frac{x}{\sqrt{\frac{2}{3}}} = (*)$

$$\int_{-1}^{1} x^2 \cdot \frac{1}{\sqrt{2}}\, dx = \left[\frac{1}{\sqrt{2}} \cdot \frac{x^3}{3}\right]_{-1}^{1} = \frac{\sqrt{2}}{3}$$

$$\int_{-1}^{1} x^2 \cdot \frac{x}{\sqrt{\frac{2}{3}}}\, dx = 0$$

$$(*) = \frac{1}{3}$$

$$\left\| x^2 - \frac{1}{3} \right\|_2^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = 2\int_0^1 x^4 - \frac{2}{3}x^2 + \frac{1}{9}\, dx =$$

$$= 2\left[\frac{x^5}{5} - \frac{2}{3}\frac{x^3}{3} + \frac{x}{9}\right]_0^1 = 2\left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right) =$$

$$= 2\left(\frac{1}{5} - \frac{1}{9}\right) = 2\,\frac{9-5}{45} = \frac{8}{45}$$

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}, \frac{x}{\sqrt{\frac{2}{3}}}, \frac{x^2 - \frac{1}{3}}{\frac{2\sqrt{2}}{3\sqrt{5}}} \right\}$$

Ora che ho base ortonormale di H, trovo proiezione di $f(x) = x^3$.

$$\Pi_f(x) = \left\langle x^3, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \left\langle x^3, \frac{x}{\sqrt{\frac{2}{3}}} \right\rangle \cdot \frac{x}{\sqrt{\frac{2}{3}}} + \left\langle x^3, \frac{x^2 - \frac{1}{3}}{\frac{2\sqrt{2}}{3\sqrt{5}}} \right\rangle \cdot \frac{x^2 - \frac{1}{3}}{\frac{2\sqrt{2}}{3\sqrt{5}}} =$$

$$= \langle x^3, 1 \rangle \cdot \frac{1}{2} + \langle x^3, x \rangle \cdot \frac{3}{2}x + \left\langle x^3, x^2 - \frac{1}{3} \right\rangle \cdot \frac{45}{8}\left(x^2 - \frac{1}{3}\right) =$$

$$\int_{-1}^{1} x^3\,dx = 0 \qquad \int_{-1}^{1}\left(x^4 - \frac{1}{3}x^2\right)dx = 0$$

$$\int_{-1}^{1} x^4\,dx = 2\cdot\left[\frac{x^5}{5}\right]_0^1 = \frac{2}{5}$$

$$\Pi_f(x) = \frac{3}{5}x$$

$$d(f,H) = d(f,\Pi_f) = \left\| x^3 - \frac{3}{5}x \right\|_2 =$$

$$= \sqrt{\int_{-1}^{1}\left(x^3 - \frac{3}{5}x\right)^2 dx} =$$

$$= \sqrt{2\int_0^1\left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right)dx} =$$

$$= \sqrt{2\left[\frac{x^7}{7} - \frac{6}{5}\frac{x^5}{5} + \frac{\cancel{9}\,3}{25}\frac{x^3}{\cancel{3}}\right]_0^1} =$$

$$= \sqrt{2\left(\frac{1}{7} - \frac{6}{25} + \frac{3}{25}\right)}$$

$$= \sqrt{2\,\frac{25-21}{7\cdot 25}} = \sqrt{\frac{8}{7\cdot 25}} = \boxed{\frac{2}{5}\frac{\sqrt{2}}{\sqrt{7}}}$$

Per casa
fare con
$H = \{\text{pol. di 4 grado}\}$
$f(x) = e^x$

ES 19
LISTA 2

Trovare serie di Fourier di $f(x) = x$ ristretta a $[-4, 4)$ e prolungata per periodicità.

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos\left(k \cdot \frac{\pi}{4} x\right) + b_k \sin\left(k \cdot \frac{\pi}{4} x\right)$$

$$a_0 = \frac{1}{4} \int_{-4}^{4} f(x)\, dx$$

$$\left.\begin{aligned} a_k &= \frac{1}{4} \int_{-4}^{4} f(x) \cos\left(k \cdot \frac{\pi}{4} x\right) dx \\ b_k &= \frac{1}{4} \int_{-4}^{4} f(x) \sin\left(k \cdot \frac{\pi}{4} x\right) dx \end{aligned}\right\} k \geq 1$$

$$\left\{ \frac{1}{2}, \cos\left(\frac{\pi}{4} x\right), \sin\left(\frac{\pi}{4} x\right), \dots, \cos\left(n \frac{\pi}{4} x\right), \sin\left(n \frac{\pi}{4} x\right) \right\}$$

$$\left\| \frac{1}{2} \right\|_2^2 = \int_{-4}^{4} \left(\frac{1}{2}\right)^2 dx = \frac{1}{4} \cdot 8 = 2$$

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$\left\| \cos\left(k \frac{\pi}{4} x\right) \right\|_2^2 = \int_{-4}^{4} \cos^2\left(k \frac{\pi}{4} x\right) dx = \int_{-4}^{4} \frac{1 + \cos\left(k \frac{\pi}{2} x\right)}{2} dx = \int_{-4}^{4} \frac{1}{2} dx = 4$$

STESSO PER SENO $\rightarrow [\dots]$

$$\left\{ \frac{1}{2\sqrt{2}}, \frac{\cos\left(\frac{\pi}{4} x\right)}{2}, \frac{\sin\left(\frac{\pi}{4} x\right)}{2}, \dots, \frac{\cos\left(n \frac{\pi}{4} x\right)}{2}, \frac{\sin\left(n \frac{\pi}{4} x\right)}{2} \right\}$$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos\left(k \cdot \frac{\pi}{4} x\right) + b_k \sin\left(k \cdot \frac{\pi}{4} x\right)$$

$$\frac{a_0}{2} = \frac{1}{2} \cdot \frac{1}{4} \int_{-4}^{4} f(x)\, dx =$$

$$= \left(\int_{-4}^{4} f(x) \cdot \frac{1}{2\sqrt{2}}\, dx \right) \cdot \frac{1}{2\sqrt{2}} = \left\langle f, \frac{1}{2\sqrt{2}} \right\rangle \cdot \frac{1}{2\sqrt{2}}$$

$$a_k \cos\left(k \frac{\pi}{4} x\right) = \left(\frac{1}{4} \int_{-4}^{4} f(x) \cos\left(k \frac{\pi}{4} x\right) dx \right) \cdot \cos\left(k \frac{\pi}{4} x\right)$$

$$= \left(\int_{-4}^{4} f(x) \cdot \frac{\cos\left(k \frac{\pi}{4} x\right)}{2}\, dx \right) \cdot \frac{\cos\left(k \frac{\pi}{4} x\right)}{2} =$$

$$= \left\langle f(x), \frac{\cos\left(k \frac{\pi}{4} x\right)}{2} \right\rangle \cdot \frac{\cos\left(k \frac{\pi}{4} x\right)}{2}$$

Se $f(x) = x$ in $[-4, 4)$ si ottiene:

$$a_0 = 0 \quad e \quad a_n = 0 \quad \forall k \geq 1$$

(perché f è dispari)

$$\underline{\underline{b_k}} = \frac{1}{4} \int_{-4}^{4} x \cdot \sin\left(k \frac{\pi}{4} x\right) dx =$$

$$= -\frac{1}{2}\frac{4}{k\pi}\int_0^4 x\cdot\left(-\frac{k\pi}{4}\sin\left(k\frac{\pi}{4}x\right)\right)dx =$$

$$= -\frac{2}{k\pi}\int_0^4 x\cdot\left(\cos\left(k\frac{\pi}{4}x\right)\right)'dx =$$

$$= -\frac{2}{k\pi}\left(\left[x\cos\left(k\frac{\pi}{4}x\right)\right]_0^4 - \cancel{\int_0^4 \cos\left(k\frac{\pi}{4}x\right)dx}^{=0}\right) =$$

$$= -\frac{2}{k\pi}\cdot 4\cos(k\pi) = \boxed{-\frac{8}{k\pi}(-1)^k}$$

$$x \sim \boxed{\sum_{k=1}^{+\infty}\frac{8\cdot(-1)^{k+1}}{k\pi}\cdot\sin\left(k\frac{\pi}{4}x\right)}$$

$f:\left[-\frac{T}{2},\frac{T}{2}\right)\to\mathbb{R}$ prolungata per periodicità

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos\left(k\frac{2\pi}{T}x\right) + b_k\sin\left(k\frac{2\pi}{T}x\right)\right)$$

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx$$

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\cos\left(k\frac{2\pi}{T}x\right)dx$$

$$b_k = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cdot \sin\left(k \frac{2\pi}{T} x\right) dx$$

ES 1 LISTA 2

Data $f: \mathbb{R} \to \mathbb{R}$ si può scrivere sempre come somma di 2 funzioni una pari e l'altra dispari nel modo seguente:

$$\boxed{\frac{f(x)+f(-x)}{2}} \qquad \boxed{\frac{f(x)-f(-x)}{2}}$$

pari $\qquad$ dispari

Per come mostrare unicità