Metodi Matematici - Ex. 6

Titolo nota

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ES. 11 LISTA 3

$$(1)\begin{cases} u_{tt}(t,x) = u_{xx}(t,x) + \underline{e^{-t}\sin(3x)} \\ f(x) = \boxed{\sin(5x)} \leftarrow 0 \\ g(x) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \qquad (t,x) \in \mathbb{R}^+ \times [0,\pi]$$

Basterà risolvere la non omogenea con $f(x) = 0$.

$$u(t,x) = \boxed{\sum_{k=1}^{+\infty} u_k(t)\cdot \sin(kx)}$$

$$u(t,x) = u(t)\sin(3x) \leftarrow$$

$$u_{tt}(t,x) = u''(t)\sin(3x)$$

$$u_{xx}(t,x) = u(t)\cdot(-9\sin(3x))$$

$$u''(t)\cancel{\sin(3x)} + 9u(t)\cancel{\sin(3x)} = e^{-t}\cancel{\sin(3x)}$$

$$\begin{cases} \underline{u''(t) + 9u(t)} = e^{-t} \\ u(t) = 0 \\ u'(t) = 0 \end{cases} \qquad \boxed{\lambda^2 + 9 = 0}$$

$$u(t) = A\cos(3t) + B\sin(3t) + \underline{u_0(t)}$$

$$\boxed{u_0(t) = K\cdot e^{-t}}$$

$$u_0'(t) = -k e^{-t}$$

$$u_0''(t) = k e^{-t}$$

$$k e^{-t} + 9 k e^{-t} = e^{-t}$$

$$\underbrace{10 k}_{=1} e^{-t} = e^{-t}$$

$$\boxed{k = \frac{1}{10}}$$

$$u(t) = A \cos(3t) + B \sin(3t) + \frac{1}{10} e^{-t}$$

$$u'(t) = -3A \sin(3t) + 3B \cos(3t) - \frac{1}{10} e^{-t}$$

$$\begin{aligned} u(0)=0 &\iff \begin{cases} A + \frac{1}{10} = 0 \iff A = -\frac{1}{10} \\ 3B - \frac{1}{10} = 0 \iff B = \frac{1}{30} \end{cases} \\ u'(0)=0 &\iff \end{aligned}$$

$$\boxed{u(t) = -\frac{1}{10} \cos(3t) + \frac{1}{30} \sin(3t) + \frac{1}{10} e^{-t}}$$

Quindi la sol di (1) con dati iniziali $f(x)=g(x)=0$ è:

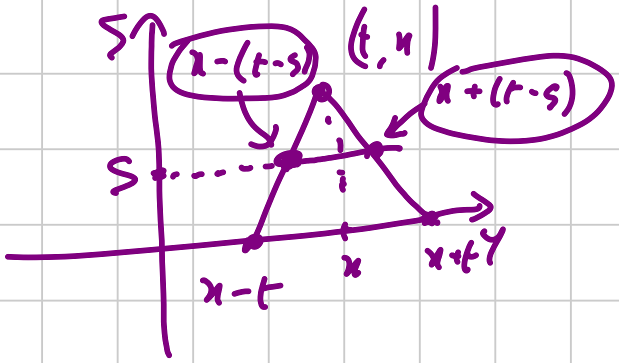
$$\boxed{u(t,x) = \left(-\frac{1}{10} \cos(3t) + \frac{1}{30} \sin(3t) + \frac{1}{10} e^{-t}\right) \sin(3x)}$$

A tale $u(t,x)$ va sommata la sol. del prob. omogeneo con dati $f(x) = \sin(5x)$ e $g(x)=0$, trovata in ES. 1 LISTA 3

$\boxed{\text{SOL. ALTERNATIVA}}$

Poiché $\underline{e^{-t}\sin(3x)}$, considerata su tutto $\mathbb{R}^+\times\mathbb{R}$ è già (dispari e periodica di per. 2π) nella x, posso risolvere (1) in tutto $\mathbb{R}^+\times\mathbb{R}$ prendendo tale $F(t,x)$ così come sta. Usando formula risolutiva, si ottiene

$$u(t,x) = \iint_{D_{t,x}} e^{-s}\sin(3y)\,ds\,dy =$$



$$= \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-s}\sin(3y)\,dy\,ds =$$

$$= \int_0^t e^{-s}\cdot\left[-\frac{1}{3}\cos(3y)\right]_{x-(t-s)}^{x+(t-s)} ds =$$

$$= -\frac{1}{3}\int_0^t e^{-s}\left(\cos\big(\underbrace{3x+3(t-s)}_{\alpha+\beta}\big) - \cos\big(\underbrace{3x-3(t-s)}_{a-\beta}\big)\right) ds =$$

$$\cancel{\cos\alpha\cos\beta} - \sin\alpha\sin\beta - \cancel{\cos\alpha\cos\beta} - \sin\alpha\sin\beta$$

$$= \frac{2}{3}\int_0^t e^{-s}\sin(3x)\sin(3(t-s))\,ds =$$

$$= \frac{2}{3}\sin(3x)\cdot\int_0^t e^{-s}\sin(3(t-s))\,ds = \ldots\ldots$$

Fare doppia integrazione per parti in

$$I(s) = \int e^{-s}\sin(3(t-s))\,ds$$

e si trova

$$I(s) = \boxed{\text{espressione in } s} - I(s)$$

$$I(s) = \frac{1}{2}\,\square \qquad [\ldots\ldots]$$

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$$(2)\begin{cases} u_{tt}(t,x) = u_{xx}(t,x) + \dfrac{\sin x}{1+t^2} & 1+(\cos t)^2 \\ f(x) = 0 & (t,x)\in\mathbb{R}^+\times[0,\pi] \\ g(x) = \boxed{\sin(5x)} \longleftrightarrow 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases}$$

$$u(t,x) = u(t)\sin x$$

$$u_{tt}(t,x) = u''(t)\sin x$$

$$u_{xx}(t,x) = -u(t)\sin x$$

Sostituendo in (2) ottengo

$$u''(t)\cancel{\sin x} + u(t)\cdot\cancel{\sin x} = \frac{\cancel{\sin x}}{1+t^2}$$

$1+(\cos t)^2$

$$\begin{cases} u''(t)+u(t) = \dfrac{1}{1+t^2} \\ u(0)=0 \\ u'(0)=0 \end{cases}$$

Metodo Var. costanti. cerco sol. del tipo:

$$u(t) = A(t)\cos t + B(t)\sin t$$

$$u'(t) = \underbrace{\boxed{A'(t)\cos t + B'(t)\sin t}}_{=0} - A(t)\sin t + B(t)\cos t$$

$$u''(t) = -A'(t)\sin t + B'(t)\cos t - A(t)\cos t - B(t)\sin t$$

$$\begin{cases} A'(t)\cos t + B'(t)\sin t = 0 \iff B'(t) = -\dfrac{\cos t}{\sin t}A'(t) \\ -A'(t)\sin t + B'(t)\cos t = \dfrac{1}{1+t^2} \end{cases}$$

$$\frac{1}{1+(\cos t)^2}$$

$$+A'(t)\sin t + \frac{\cos^2 t}{\sin t}A'(t) = -\frac{1}{1+t^2}$$

$$\frac{\sin^2 t + \cos^2 t}{\sin t}A'(t) = -\frac{1}{1+t^2}$$

$$\begin{cases} A'(t) = -\dfrac{\sin t}{1+(\cos t)^2} \\ B'(t) = \dfrac{\cos t}{1+(\cos t)^2} \end{cases}$$

$$A(t) = \int_0^t \frac{-\sin s}{1+(\cos s)^2}\, ds = \int_0^t \frac{1}{1+(\cos s)^2}\cdot(\cos s)'\, ds =$$

$$= \int_1^{\cos t} \frac{1}{1+u^2}\, du = \boxed{\arctan(\cos t) - \frac{\pi}{4}}$$

$$B(t) = \int_0^t \frac{\cos s}{1+(\cos s)^2}\, ds = \int_0^t \frac{1}{2-(\sin s)^2}\,(\sin s)'\, ds =$$

$$= \int_{\sin 0}^{\sin t} \frac{1}{2-u^2}\, du =$$

$$\frac{1}{2\sqrt{2}}\, \frac{\sqrt{2}+u+\sqrt{2}-u}{(\sqrt{2}+u)(\sqrt{2}-u)} = \boxed{\frac{1}{2\sqrt{2}}\, \frac{1}{\sqrt{2}-u} + \frac{1}{2\sqrt{2}}\, \frac{1}{\sqrt{2}+u}}$$

$$= \frac{1}{2\sqrt{2}} \int_{\sin 0}^{\sin t} \left(-\frac{1}{u-\sqrt{2}} + \frac{1}{u+\sqrt{2}}\right) du = \frac{1}{2\sqrt{2}} \left[\ln\left|\frac{u+\sqrt{2}}{u-\sqrt{2}}\right|\right]_0^{\sin t} =$$

$$= \boxed{\frac{1}{2\sqrt{2}} \ln\left(\frac{\sin t+\sqrt{2}}{\sqrt{2}-\sin t}\right)}$$

quelli appena trovati

$$u(t) = A(t)\cos t + B(t)\sin t$$

$$u(t,x) = \boxed{u(t)\cdot \sin x}$$