Metodi Matematici - Ex. 11

Titolo nota

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$\boxed{\text{ES. 1}}$ $T = \sum_{k=-\infty}^{+\infty} \delta_k$ mostrare che è periodica di periodo 1.

$\boxed{\text{Def.}}$ $T \in \mathcal{D}'(\mathbb{R})$ è periodica di periodo T se $\forall \varphi \in \mathcal{D}(\mathbb{R})$ $T(\varphi(x)) = T(\varphi(x+T))$

Dobbiamo mostrare che $\forall \varphi \in \mathcal{D}(\mathbb{R})$

$$T(\varphi(x)) \overset{?}{=} T(\varphi(x+1))$$

$$T(\varphi(x)) = \sum_{k=-\infty}^{+\infty} \varphi(k) \underset{n=k-1}{\overset{\uparrow}{=}} \sum_{n=-\infty}^{+\infty} \varphi(n+1)$$

$$=$$

$$T(\varphi(x+1)) = \sum_{k=-\infty}^{+\infty} \varphi(k+1)$$

$\boxed{\text{ES 2}}$ Verificare che T è dispari, dove

$$T : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$$

$$\varphi \longmapsto \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx.$$

Verificata scorsa volta T è ben definita e lineare.

Mostriamo che è continua ① e dispari ②:

① $\underbrace{\varphi_n \xrightarrow{\mathcal{D}(\mathbb{R})} 0}_{(*)} \overset{?}{\Rightarrow} T(\varphi_n) \to 0$

$K \supset \operatorname{supp} \varphi_n \ \forall n$

Prendo $a>0$ t.c. $K \subset [-a,a]$

$$T(\varphi_n) = \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{\varphi_n(x)}{x}\, dx = \lim_{\varepsilon \to 0^+} \int_{\substack{|x|>\varepsilon \\ |x|<a}} \frac{\varphi_n(x)}{x}\, dx =$$

compreso tra 0 e x

$$= \lim_{\varepsilon \to 0^+} \int_{\substack{|x|>\varepsilon \\ |x|<a}} \frac{\varphi_n(x) - \varphi(0)}{x - 0}\, dx = \lim_{\varepsilon \to 0^+} \int_{\substack{|x|>\varepsilon \\ |x|<a}} \varphi_n'(c(x))\, dx$$

Perché $\forall \varepsilon > 0 \quad \displaystyle\int_{\substack{|x|>\varepsilon \\ |x|<a}} \frac{\varphi(0)}{x}\, dx = 0$

Quindi

$$|T(\varphi_n)| \leq \lim_{\varepsilon \to 0^+} \int_{\substack{|x|>\varepsilon \\ |x|<a}} |\varphi_n'(c(x))|\, dx \leq$$

$$\leq \lim_{\varepsilon \to 0^+} \int_{\substack{|x|>\varepsilon \\ |x|<a}} \|\varphi_n'\|_{L^\infty}\, dx =$$

$$= \lim_{\varepsilon \to 0^+} \|\varphi_n'\|_{L^\infty} \cdot (2a - 2\varepsilon) =$$

$$= \|\varphi_n'\|_{L^\infty} \cdot 2a$$

$$0 \le \underbrace{|T(\varphi_n)|}_{\downarrow \atop 0} \le 2a \cdot \underbrace{\|\varphi_n'\|_{L^\infty}}_{\downarrow \leftarrow (*) \atop 0}$$

Quindi T è continua.

(2) T è dispari cioè $\forall \varphi \in \mathcal{D}(\mathbb{R})$ $T(\varphi(x)) = \underline{-T(\varphi(-x))}$

$$-T(\varphi(-x)) = -\lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(-x)}{x}\, dx =$$

$y = -x$

$$= -\lim_{\varepsilon \to 0^+} \left(\int_{-a}^{-\varepsilon} \frac{\varphi(-x)}{-x} (-1)\, dx + \int_{\varepsilon}^{a} \frac{\varphi(-x)}{-x} (-1)\, dx \right) =$$

$$= -\lim_{\varepsilon \to 0^+} \left(\int_{a}^{\varepsilon} \frac{\varphi(y)}{y}\, dy + \int_{-\varepsilon}^{-a} \frac{\varphi(y)}{y}\, dy \right) =$$

$$= \lim_{\varepsilon \to 0^+} \left(\int_{\varepsilon}^{a} \frac{\varphi(y)}{y}\, dy + \int_{-a}^{-\varepsilon} \frac{\varphi(y)}{y}\, dy \right) =$$

$$= \lim_{\varepsilon \to 0^+} \int_{\substack{|x|>\varepsilon \\ |x|<a}} \frac{\varphi(y)}{y}\, dy =$$

$$= \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(y)}{y}\, dy = T(\varphi(x))$$

$\boxed{\text{ES } 3}$ $\Big(\underbrace{(T \text{ è pari})}_{(*)}$ e $\underbrace{(\varphi \text{ dispari})}_{(**)} \Big) \Longrightarrow T(\varphi) = 0$

$(*)$: $T(\varphi(x)) = T(\varphi(-x))$

$(**)$: $\varphi(x) = -\varphi(-x)$

$\forall \varphi \in \mathcal{D}(\mathbb{R})$ con φ dispari si ha:

$$\left.\begin{aligned} T(\varphi(x)) &\overset{(*)}{=} T(\varphi(-x)) \\ \overset{(**)}{\parallel} \quad & \\ T(-\varphi(-x)) &= -T(\varphi(-x)) \end{aligned}\right\} \Longrightarrow T(\varphi(-x)) = -T(\varphi(-x))$$

$$\Downarrow$$

$$T(\varphi(-x)) = 0$$

$$\Downarrow$$

$$T(-\varphi(x)) = 0$$

$$\Downarrow$$

$$-T(\varphi(x)) = 0$$

$$\Downarrow$$

$$T(\varphi(x)) = 0$$

$\boxed{\text{ES 4}}$ fatto in classe

$\boxed{\text{ES. 5}}$ Mostrare che se $T \in \mathcal{D}'(\mathbb{R})$ e $f \in C^\infty(\mathbb{R})$ allora

$$(fT)' = f'T + fT'.$$

$\boxed{\text{Sol}}$ Basta far vedere che $\forall \varphi \in \mathcal{D}(\mathbb{R})$ si ha

$$\left((fT)'\right)(\varphi(x)) \overset{?}{=} \left(f'T + fT'\right)(\varphi(x))$$

$$(fT)'(\varphi(x)) = -(fT)(\varphi'(x)) = -T(f(x)\varphi'(x)) = T(-f(x)\cdot\varphi'(x))$$

$$(f'T + fT')(\varphi(x)) = (f'T)(\varphi(x)) + (fT')(\varphi(x)) =$$

$$= T(f'(x)\varphi(x)) + T'(f(x)\varphi(x)) =$$

$$= T(f'(x)\varphi(x)) - T((f(x)\varphi(x))') =$$

$$= T(f'(x)\varphi(x)) - T(f'(x)\varphi(x) + f(x)\varphi'(x)) =$$

$$= T(\cancel{f'(x)\varphi(x)} - \cancel{f'(x)\varphi(x)} - f(x)\varphi'(x)) =$$

$$= T(-f(x)\varphi'(x))$$

Quindi $(fT)'$ e $f'T + fT'$ sono la stessa distribuzione.

$\boxed{\text{ES6}}$ $\forall a>0$ definiamo $T_a : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$

$$\varphi \longmapsto \int_{-\infty}^{+\infty} \frac{x}{a^2+x^2}\,\varphi(x)\,dx$$

Mostrare che $\lim_{a\to 0^+} T_a \overset{\text{nel senso delle distribuzioni}}{=} T$

dove $T(\varphi) = \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx$

$\boxed{\text{SOL}}$ Dobbiamo mostrare che $\forall \varphi \in \mathcal{D}(\mathbb{R})$

$$\lim_{a\to 0^+} T_a(\varphi) = T(\varphi)$$

Prendo $b>0$ t.c. supp $\varphi \subset [-b,b]$

$$T_a(\varphi) = \int_{-\infty}^{+\infty} \frac{x}{a^2+x^2}\,\varphi(x)\,dx = \int_{-b}^{b} \frac{x}{a^2+x^2}\,\varphi(x)\,dx =$$

$$= \int_{-b}^{b} \frac{x^2}{a^2+x^2}\left(\frac{\varphi(x)-\varphi(0)}{x-0}\right)dx =$$

$$T(\varphi) = \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx =$$

$$= \lim_{\varepsilon \to 0^+} \int_{\substack{|x|>\varepsilon \\ |x|<b}} \boxed{\frac{\varphi(x)-\varphi(0)}{x-0}}\, dx = \int_{-b}^{b} \frac{\varphi(x)-\varphi(0)}{x-0}\, dx$$

la funzione ha una disc. eliminabile per $x=0$ perché:

$$\psi(x) = \begin{cases} \dfrac{\varphi(x)-\varphi(0)}{x-0} & x \neq 0 \\ \varphi'(0) & x = 0 \end{cases}$$

Quindi

$$|T_a(\varphi) - T(\varphi)| = \left| \int_{-b}^{b} \frac{x^2}{a^2+x^2} \cdot \frac{\varphi(x)-\varphi(0)}{x-0}\, dx - \int_{-b}^{b} \frac{\varphi(x)-\varphi(0)}{x-0}\, dx \right|$$

$$= \left| \int_{-b}^{b} \left(\frac{x^2}{a^2+x^2} - 1 \right) \frac{\varphi(x)-\varphi(0)}{x-0}\, dx \right| =$$

con $c(x)$ compreso tra 0 e x

$$= \left| \int_{-b}^{b} \frac{-a^2}{a^2+x^2}\, \varphi'(c(x))\, dx \right| \leq$$

$$\leq \int_{-b}^{b} \frac{a^2}{a^2+x^2} \cdot |\varphi'(c(x))|\, dx \leq$$

$$\leq \int_{-b}^{b} \frac{a^2}{a^2+x^2} \cdot \|\varphi'\|_{L^\infty}\, dx =$$

$$= a^2 \, \|\varphi'\|_{L^\infty} \cdot \int_{-b}^{b} \frac{1}{a^2+x^2}\, dx \leq$$

$$\leq 2b \cdot a^2 \, \|\varphi'\|_{L^\infty}$$

$$\underbrace{0}_{\downarrow \, 0} \leq \underbrace{\left| T_a(\varphi) - T(\varphi) \right|}_{\downarrow \, 0} \leq \underbrace{\overset{\to 0}{a^2} \cdot 2b \cdot \|\varphi'\|_{L^\infty}}_{\downarrow \, 0}$$

Quindi $T_a(\varphi) \to T(\varphi)$ per $a \to 0^+$