Metodi Matematici - Lez. 7

Titolo nota

16 ottobre 2017 (14:00-15.45) - docente: Prof. **Emanuele Callegari** - Università di Roma ***Tor Vergata***

SERIE DI FOURIER

Ult. Teo. $S_n(x) \to f$ in $L^2(-\pi, \pi)$ se $f \in L^2$

dove $S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx) \right)$

con $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$\left.\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx \end{aligned}\right\} \quad k = 1, 2, \dots$$

Definiamo serie di Fourier di f come la serie di funzioni avente $S_n(x)$ come succ. di somme finite, cioè:

$$\frac{a_0}{2} + \sum_{k=1}^{+\infty} \left(a_k \cos(kx) + b_k \sin(kx) \right)$$

$\boxed{\text{OSS. 1}}$ $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\}$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx) \right)$$

$$\frac{a_0}{2} = \frac{1}{2} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\pi}^{\pi} f(x) \cdot \frac{1}{\sqrt{2\pi}}\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle$$

$$a_k \cos(kx) = \cos(kx) \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx =$$

$$= \frac{\cos(kx)}{\sqrt{\pi}} \cdot \int_{-\pi}^{\pi} f(x) \frac{\cos(kx)}{\sqrt{\pi}}\, dx =$$

$$= \left\langle f, \frac{\cos(kx)}{\sqrt{\pi}} \right\rangle \cdot \frac{\cos(kx)}{\sqrt{\pi}}$$

$$b_k \sin(kx) = \cdots\cdots = \left\langle f, \frac{\sin(kx)}{\sqrt{\pi}} \right\rangle \cdot \frac{\sin(kx)}{\sqrt{\pi}}$$

$\boxed{\text{OSS 2}}$ $\boxed{\text{Din. di Bessel}}$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx) \right)$$

$$\| S_n(x) \|_2^2 = \left\| \frac{a_0}{2} \right\|_2^2 + \sum_{k=1}^{n} \left(\| a_k \cos(kx) \|_2^2 + \| b_k \sin(kx) \|_2^2 \right)$$

$$= \left\| \frac{a_0\sqrt{\pi}}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} \right\|_2^2 + \sum_{k=1}^{n} \left(\left\| \sqrt{\pi}\, a_k \frac{\cos(kx)}{\sqrt{\pi}} \right\|_2^2 + \left\| \sqrt{\pi}\, b_k \frac{\sin(kx)}{\sqrt{\pi}} \right\|_2^2 \right) =$$

PER IL T. DELLA PROIEZIONE

$$= \frac{a_0^2\,\pi}{2} + \sum_{k=1}^{n} \pi a_k^2 + \pi b_k^2 \le \int_{-\pi}^{\pi} (f(x))^2\, dx$$

$$\boxed{\frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + b_k^2 \le \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\, dx}$$

OSS 3 $\quad S_n(x) \xrightarrow{L^2} f$ cioè $\|S_n - f\|_{L^2} \to 0$

$$\underbrace{\|f\|_2^2} = \|(f - S_n) + S_n\|_2^2 = \underbrace{\|f - S_n\|_2^2}_{\downarrow\, 0} + \underbrace{\|S_n\|_2^2}_{\downarrow\, \|f\|_2^2}$$

$$\underbrace{\|S_n\|_2^2 \to \|f\|_2^2}$$

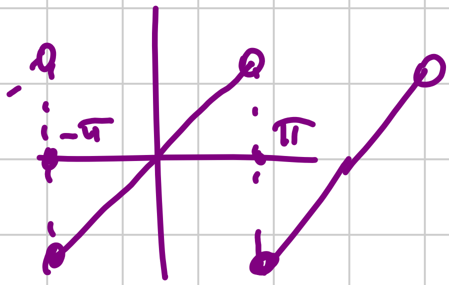
$$\frac{a_0^2}{2} + \sum_{k=1}^{+\infty} (a_k^2 + b_k^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\, dx$$

IDENTITÀ DI PARSEVAL

ES. 4 LISTA 2

$f(x) = x$ in $[-\pi, \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\, dx = 0$$

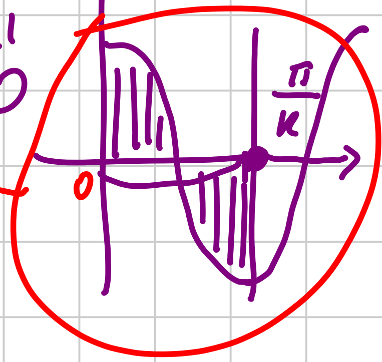


$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(kx)\,dx = \dots = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cdot\left(-\frac{\cos(kx)}{k}\right)' dx$$

$$= \frac{2}{\pi}\left(\left[-x\,\frac{\cos(kx)}{k}\right]_0^{\pi} + \frac{1}{k}\underbrace{\int_0^{\pi}\cos(kx)\,dx}_{=0}\right) =$$

$$= \frac{2}{\cancel{\pi}}\left(-\cancel{\pi}\,\frac{\overbrace{\cos(k\pi)}^{(-1)^k}}{k}\right) =$$

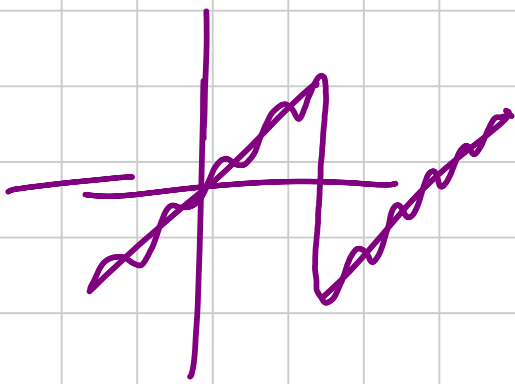


$$= \frac{2}{k}\cdot(-1)^{k+1}$$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos(kx) + b_k\sin(kx)\right) =$$

$$= \sum_{k=1}^{n}\frac{2}{k}(-1)^{k+1}\sin(kx)$$

$$x \sim \sum_{k=1}^{+\infty}\frac{2}{k}(-1)^{k+1}\sin(kx)$$



ES 13 LISTA 2

$$\sum_{k=1}^{+\infty} b_k^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2}{3}\pi^2$$

ID. PARS.

$$\parallel$$

$$\sum_{k=1}^{+\infty} \frac{4}{k^2}$$

$$4\sum_{k=1}^{+\infty}\frac{1}{k^2} = \frac{2}{3}\pi^2$$

$$\boxed{\sum_{k=1}^{+\infty}\frac{1}{k^2} = \frac{\pi^2}{6}}$$

ES. 5 LISTA 2

$$f(x) = |x|$$

$\forall k \qquad b_k = 0$ perché $|x|$ è pari e quindi $|x|\cdot\sin(kx)$ è dispari.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\cancel{2}}{\cancel{\pi}}\cdot\frac{\pi^{\cancel{2}}}{\cancel{2}} = \pi$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(kx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\left(\frac{\sin(kx)}{k}\right)'dx =$$

$$= \frac{2}{\pi}\left(\underbrace{\cancel{\left[x\frac{\sin(kx)}{k}\right]_0^{\pi}}}_{=0} - \frac{1}{k}\int_0^{\pi}\sin(kx)\,dx\right) =$$

$$= + \frac{2}{k^2\pi}\int_0^{\pi} -\sin(kx)\cdot k \, dx =$$

$$= + \frac{2}{k^2\pi}\Big[\cos(kx)\Big]_0^{\pi} = \frac{2}{k^2\pi}\Big(\overbrace{\cos(k\pi)}^{(-1)^k} - \overset{1}{\cos 0}\Big)$$

$$= \frac{2\left((-1)^k - 1\right)}{k^2\pi} = \begin{cases} 0 & \text{se } k \text{ pari} \\ \dfrac{-4}{k^2\pi} & \text{se } k \text{ dispari} \end{cases}$$

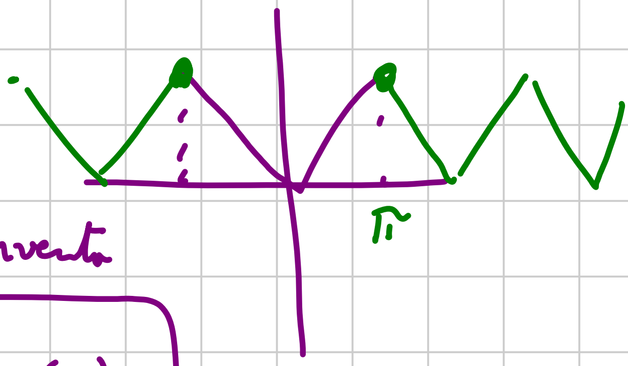
$$S_n(x) = \frac{\overset{\pi}{a_0}}{2} + \sum_{k=1}^{n}\Big(a_k \cos(kx) + \overset{0}{b_k}\sin(kx)\Big)$$

k: $2m+1 \in \mathbb{N}$

$$|x| \sim \frac{\pi}{2} - 4\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2\pi}\cdot\cos((2m+1)x)$$

ES 14 LISTA 2

$$\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2} = ?$$



PER $x = \pi$ lo sviluppo di $|x|$ diventa

$$\boxed{\pi = \frac{\pi}{2} + 4\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2\cdot\pi}\cdot(+1)}$$

HO CONVERGENZA PERCHÈ $f(x)$ È REGOLARE A TRATTI

$$\frac{\pi}{2} = \frac{4}{\pi}\sum_{m=0}^{+\infty}\frac{1}{(2m+1)^2} \qquad \sum_{m=0}^{+\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$$

Controprova

So già che $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

$$\frac{1}{4}\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{24}$$

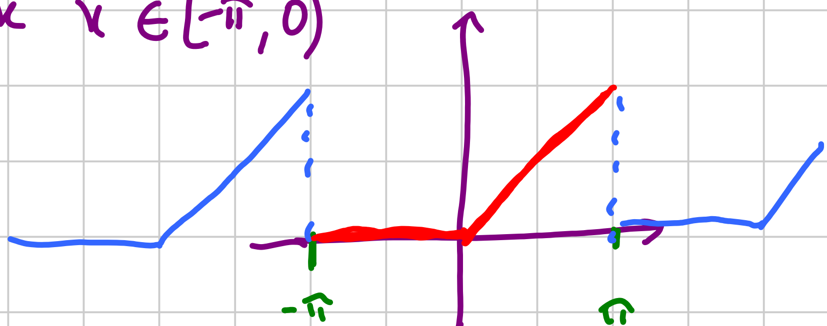
CIOÈ: $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{24}$

$$\boxed{\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2}} = \sum_{k=1}^{+\infty} \frac{1}{k^2} - \sum_{k=1}^{+\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} =$$

$$= \pi^2 \frac{4-1}{24} = \frac{3}{24}\pi^2 = \boxed{\frac{\pi^2}{8}}$$

ES. FUORI LISTA

$$f(x) = \begin{cases} x & \text{se } x \in [0, \pi] \\ 0 & \text{se } x \in [-\pi, 0) \end{cases}$$



$$f(x) = \frac{x + |x|}{2} = \begin{cases} \frac{\cancel{2}x}{\cancel{2}} & (x > 0) \\ 0 & (x < 0) \end{cases}$$

$$h(x) \sim \frac{a_0}{2} + \sum_{k=1}^{+\infty} a_k \cos(kx) + b_k \sin(kx)$$

$$g(x) \sim \frac{\alpha_0}{2} + \sum_{k=1}^{+\infty} \alpha_k \cos(kx) + \beta_k \sin(kx)$$

$$\boxed{A, B \in \mathbb{R}}$$

$$A h(x) + B g(x) \sim \frac{A a_0 + B \alpha_0}{2} + \sum_{k=1}^{+\infty} \underbrace{\left(A a_k + B \alpha_k\right)} \cos(kx) + \left(A b_k + B \beta_k\right) \sin(kx)$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \left(A h(x) + B g(x)\right) \cos(kx)\, dx =$$

$$= A \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} h(x) \cos(kx)\, dx + B \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(kx)\, dx =$$

$$= A \cdot a_k + B \alpha_k$$

quindi per $f(x) = \frac{1}{2} x + \frac{1}{2} |x|$ basta prendere i coeff. di Fourier già trovati [...]