Metodi Matematici - Lez. 10

Titolo nota

24 ottobre 2017 (9:30-11.15) - docente: Prof. **Emanuele Callegari** - Università di Roma ***Tor Vergata***

EQ. ONDE (ALTRO MODO)

Teorema Date $f \in C^2(\mathbb{R})$ e $g \in C^1(\mathbb{R})$ allora $\exists!$ sol. $u(t,x)$ del prob.

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

ed è data dalla formula:

$$(*) \quad u(t,x) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

$$\int_0^{x+ct} g(y) - \int_0^{x-ct} g(y)$$

DIM **I° passo** $u(t,x)$ data da $(*)$ è davvero sol.

$$u_t(t,x) = \frac{f'(x+ct)-f'(x-ct)}{2}\cdot c + \frac{1}{2\cancel{c}}\left(g(x+ct)\cdot\cancel{c} + g(x-ct)\cdot\cancel{c}\right) =$$

$$= \frac{f'(x+ct)-f'(x-ct)}{2}\cdot c + \frac{g(x+ct)+g(x-ct)}{2}$$

$$\boxed{u_{tt}(t,x) = \frac{f''(x+ct)+f''(x-ct)}{2}\,c^2 + \frac{g'(x+ct)-g'(x-ct)}{2}\cdot c}$$

$$u_x(t,x) = \frac{f'(x+ct)+f'(x-ct)}{2} + \frac{1}{2c}\cdot\left(g(x+ct)-g(x-ct)\right)$$

$$u_{xx}(t,x) = \frac{f''(x+ct)+f''(x+ct)}{2} + \frac{g'(x+ct)-g'(x-ct)}{2c}$$

$\boxed{\text{II° Passo}}$ $\boxed{\text{unicità}}$

Sia $\boxed{u}$ sol. di $u_{tt} = c^2 u_{xx}$

$$\begin{cases} \eta = x+ct \\ \xi = x-ct \end{cases} \qquad \begin{cases} x = \frac{\eta+\xi}{2} \\ t = \frac{\eta-\xi}{2c} \end{cases}$$

$$v(\eta,\xi) = u\left(\frac{\eta-\xi}{2c}, \frac{\eta+\xi}{2}\right)$$

$$v_\eta(\eta,\xi) = u_t\left(\frac{\eta-\xi}{2c}, \frac{\eta+\xi}{2}\right)\cdot\frac{1}{2c} + u_x(\ldots,\ldots)\frac{1}{2} =$$

$$= \frac{1}{2c}u_t + \frac{1}{2}u_x$$

$$v_{\eta\xi}(\eta,\xi) = \frac{1}{2c}\cdot\left(u_{tt} \quad \ldots\ldots\right) = [\ldots] = 0$$

$$v_{\eta\eta}(\eta,\xi) = 0 \Rightarrow \int_0^{\eta_0} v_{\eta\xi}(\eta,\xi)\,d\eta = 0 \Rightarrow v_\xi(\eta_0,\xi) - v_\xi(0,\xi) = 0 \Rightarrow$$

$$\Rightarrow \int_0^{\xi} v_\xi(\eta_0,\xi) - v_\xi(0,\xi)\,d\xi = 0 \Rightarrow v(\eta_0,\xi_0) - v(\eta_0,0) - v(0,\xi_0) + v(0,0) = 0$$

$$\Rightarrow \boxed{v(\eta_0,\xi_0)} = \overbrace{v(\eta_0,0)}^{\alpha(\eta_0)} + \overbrace{v(0,\xi_0) - v(0,0)}^{\beta(\xi_0)} \quad \forall \eta_0,\xi_0$$

$$v(\eta,\xi)=\alpha(\eta)+\beta(\xi)$$

$$u(t,x)=\alpha(x+ct)+\beta(x-ct)$$

$$u_1(t,x)=\alpha_1(x+ct)+\beta_1(x-ct)$$

$$u_2(t,x)=\alpha_2(x+ct)+\beta_2(x-ct)$$

$$u_1(0,x)=f(x)=u_2(0,x)$$

$$\alpha_1(x)+\beta_1(x)=\alpha_2(x)+\beta_2(x)$$

$$u_{1t}(0,x)=g(x)=u_{2t}(0,x)=$$

$$c\,\alpha_1'(x)-c\,\beta_1'(x)=c\,\alpha_2'(x)-c\,\beta_2'(x)$$

$$\begin{cases}\alpha_1'(x)+\beta_1'(x)=f'(x)\\ \alpha_1'(x)-\beta_1'(x)=\frac{1}{c}g(x)\end{cases}\Rightarrow\begin{cases}\alpha_1'(x)=\frac{1}{2}f'(x)+\frac{1}{2c}g(x)\\ \beta_1'(x)=\frac{1}{2}f'(x)-\frac{1}{2c}g(x)\end{cases}\quad 1$$

$$\begin{cases}\alpha_2'(x)+\beta_2'(x)=f'(x)\\ \alpha_2'(x)-\beta_2'(x)=\frac{1}{c}g(x)\end{cases}\Rightarrow\begin{cases}\alpha_2'(x)=\frac{1}{2}f'(x)+\frac{1}{2c}g(x)\\ \beta_2'(x)=\frac{1}{2}f'(x)-\frac{1}{2c}g(x)\end{cases}\quad 2$$

$$\alpha_1(x)=C_1+\frac{1}{2}f(x)+\frac{1}{2c}\int_0^x g(y)\,dy$$

$$\beta_1(x)=C_2+\frac{1}{2}f(x)-\frac{1}{2c}\int_0^x g(y)\,dy$$

$$u_1(t,x)=\alpha_1(x+ct)+\beta_1(x-ct)=$$

$$= C_1 + \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(y)\,dy + C_2 + \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(y)\,dy =$$

$$= C_1 + C_2 + \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

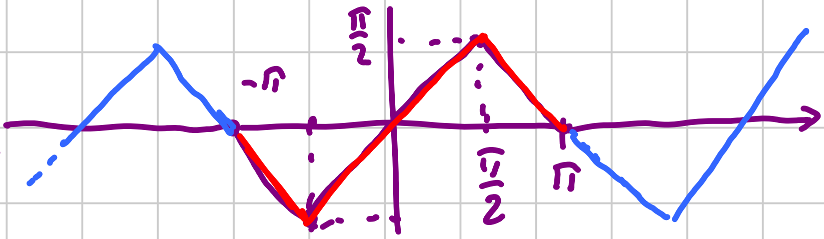
$$u_1(0,x) = f(x) \Rightarrow C_1 + C_2 = 0$$

$$u_1(t,x) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

Stesso posso fare per u_2, quindi trovo $u_1 = u_2$.

ESEMPIO

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,x) = f(x) \\ u_t(0,x) = 0 \end{cases}$$



$$u(t,x) = \frac{f(x+ct)+f(x-ct)}{2}$$