Metodi Matematici - Lez. 14

Titolo nota

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TRASF. DI FOURIER (CONTINUA ...)

⑥ $f(x) \to f(x-x_0)$

$$\left[(f(x-x_0))\right]^{\wedge}(\lambda) = e^{-ix_0\lambda}\hat{f}(\lambda)$$

$$(f(x-x_0))^{\wedge}(\lambda) = \int_{-\infty}^{+\infty} f(t-x_0)\, e^{-it\lambda}\, dt =$$

$$= \int_{-\infty}^{+\infty} f(t-x_0)\, e^{-i(t-x_0)\lambda} \cdot e^{-ix_0\lambda}\, dt$$

$$= e^{-ix_0\lambda} \cdot \int_{-\infty}^{+\infty} f(t-x_0)\, e^{-i(t-x_0)\lambda} \cdot (t-x_0)'\, dt$$

$$= e^{-ix_0\lambda} \int_{-\infty}^{+\infty} f(s)\, e^{-is\lambda}\, ds =$$

$$= e^{-ix_0\lambda}\, \hat{f}(\lambda)$$

⑦ $f(x) \longrightarrow e^{i\lambda_0 x} f(x)$

$$\left(e^{i\lambda_0 x} f(x)\right)^{\wedge}(\lambda) = \hat{f}(\lambda - \lambda_0)$$

$$\left(e^{i\lambda_0 x} \cdot f(x)\right)^{\wedge}(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda_0 t} f(t) \cdot e^{-i\lambda t}\, dt$$

$$= \int_{-\infty}^{+\infty} f(t)\, e^{-i\overbrace{(\lambda-\lambda_0)}t}\, dt =$$

$$= \hat{f}(\lambda-\lambda_0)$$

8) $f \to \hat{f}$ $\quad\boxed{f \in L^1 \quad f' \in L^1}$

$f' \to\ ?$

$$\boxed{(f')^\wedge(\lambda)} = \int_{-\infty}^{+\infty} f'(t)\, e^{-i\lambda t}\, dt =$$

$$= \int_{-\infty}^{+\infty} f(t) \left(e^{-i\lambda t}\right)' dt =$$

$$= -i\lambda \int_{-\infty}^{+\infty} f(t)\, e^{-i\lambda t}\, dt =$$

$\boxed{\begin{array}{c} f, \dots, f^{(n)} \\ \cap \\ L^1(\mathbb{R}) \end{array}}$

$$= \boxed{-i\lambda\ \hat{f}(\lambda)}$$

$$\boxed{\left(f^{(n)}\right)^\wedge(\lambda) = (-i\lambda)^n \hat{f}(\lambda)}$$

9) $f(x) \in L^1$ e $x f(x) \in L^1$

$\boxed{\begin{array}{c} f, xf, \dots, x^n f \in L^1 \\ \left((-ix)^n f(x)\right)^\wedge(\lambda) = \left(\hat{f}(\lambda)\right)^{(n)} \end{array}}$

$f(x) \to \hat{f}(\lambda)$

$$\left(-ix\, f(x)\right)^\wedge(\lambda) = \left(\hat{f}(\lambda)\right)'$$

$$\left(-i\, x f(x)\right)^\wedge(\lambda) = \int_{-\infty}^{+\infty} -it\, f(t)\, e^{-i\lambda t}\, dt =$$

$$= \int_{-\infty}^{+\infty} f(t) \cdot \left(e^{-i\lambda t}\right)_\lambda dt =$$

$\boxed{\text{OSS}}$

$$= \left(\int_{-\infty}^{+\infty} f(t)\, e^{-i\lambda t}\, dt \right)_\lambda =$$

$$= \left(\hat{f}(\lambda)\right)'$$

$\boxed{\text{OSS.}}$

$$\lim_{h\to 0} \frac{1}{h}\left(\int_{-\infty}^{+\infty} f(t)\, e^{-i(\lambda+h)t}\, dt - \int_{-\infty}^{+\infty} f(t)\, e^{-i\lambda t}\, dt \right)$$

$$= \lim_{h\to 0} \int_{-\infty}^{+\infty} f(t) \left(\frac{e^{-i(\lambda+h)t} - e^{-i\lambda t}}{h} \right) dt =$$

per T. conv. dominata

grazie al fatto che $t f(t) \in L^1$

$$= \int_{-\infty}^{+\infty} f(t) \left(e^{-i\lambda t}\right)_\lambda dt \quad \boxed{\text{OK}}$$

ES. Trasf. Fourier

$$f(x) = \chi_{[a,b]}(x)$$

$$\hat{f}(\lambda) = \int_{-\infty}^{+\infty} \chi_{[a,b]}(t)\, e^{-i\lambda t}\, dt =$$

$$\text{se } \lambda \neq 0 \quad = \int_a^b 1 \cdot e^{-i\lambda t}\, dt = \left[\frac{e^{-i\lambda t}}{-i\lambda} \right]_a^b =$$

$$= \frac{e^{-i\lambda b} - e^{-i\lambda a}}{-i\lambda}$$

$$\hat{f}(\lambda) = \begin{cases} b-a & \text{se } \lambda = 0 \\ \dfrac{e^{-i\lambda b} - e^{-i\lambda a}}{-i\lambda} & \text{se } \lambda \neq 0 \end{cases}$$

$$\lim_{\lambda \to 0} \frac{e^{-i\lambda b} - e^{-i\lambda a}}{-i\lambda} =$$

$$= \lim_{\lambda \to 0} \underbrace{\boxed{e^{-i\lambda a}}}_{1} \cdot \frac{e^{-i\lambda \overline{(b-a)}} - 1}{-i\lambda} = 1 \cdot (b-a)$$

ES. 2 $\quad \boxed{f(x) = e^{-|x|}}$

$$\hat{f}(\lambda) = \int_{-\infty}^{+\infty} e^{-|t|} e^{-i\lambda t}\, dt =$$

$$= \int_{-\infty}^{0} e^{t} \cdot e^{-i\lambda t}\, dt + \int_0^{+\infty} e^{-t} e^{-i\lambda t}\, dt =$$

$$= \int_{-\infty}^{0} e^{(1-\lambda i)t}\,dt + \int_{0}^{+\infty} e^{-(1+\lambda i)t}\,dt =$$

$$= \left[\frac{e^{(1-\lambda i)t}}{1-\lambda i}\right]_{-\infty}^{0} + \left[\frac{e^{-(1+\lambda i)t}}{-1-\lambda i}\right]_{0}^{+\infty} =$$

$$= \frac{1}{1-\lambda i} + \frac{1}{1+\lambda i} = \frac{1 \cancel{+\lambda i} + 1 \cancel{-\lambda i}}{1+\lambda^2} =$$

$$= \frac{2}{1+\lambda^2}$$

$$\underbrace{\boxed{e^{t}} \cdot \boxed{e^{-i\lambda t}}}_{\downarrow\ 0} \leftarrow \text{per } t \to -\infty$$