# Metodi Matematici - Lez. 15

Titolo nota

13 Novembre 2017 (14:00-15.45) - docente: Prof. **Emanuele Callegari** - Università di Roma ***Tor Vergata***

$\boxed{\text{Teorema}}$ Data $f \in L^1(\mathbb{R}) \cap C^1_{\text{tratti}}(\mathbb{R})$ allora $\forall x \in \mathbb{R}$

$$(1) \qquad \frac{f(x^+)+f(x^-)}{2} = \frac{1}{2\pi} \underbrace{v.p. \int_{-\infty}^{+\infty} \hat{f}(\lambda) e^{i\lambda x} \, d\lambda}$$

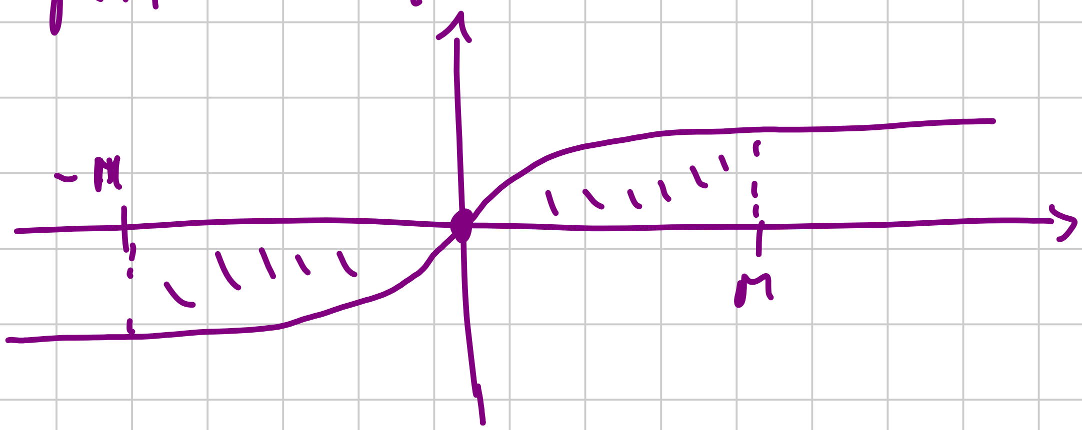
$$= \lim_{M \to +\infty} \int_{-M}^{M} \hat{f}(\lambda) e^{i\lambda x} \, d\lambda$$

$\boxed{\text{Oss.}}\boxed{1}$

Se $\int_{-M}^{M} f(t)\,dt \to$ limite finito per $M \to +\infty$

non è detto che $\int_{-\infty}^{+\infty} f(t)\,dt$ esiste.

Es. $f(t) = \operatorname{arctan} t$




$$\forall M \in \mathbb{R}^+ \quad \int_{-M}^{M} \operatorname{arctan} t \, dt = 0$$

$\boxed{\text{OSS 2}}$ se $\hat{f} \in L^1(\mathbb{R})$ allora in (1) sempre v.p.

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$\boxed{\text{Iterazione di } \wedge}$

(2) Ipotesi oltre a $f \in L^1(\mathbb{R}) \cap C^1_{\text{tratti}}(\mathbb{R})$
prendiamo $\hat{f} \in L^1(\mathbb{R})$
e "normalizziamo" la $f$ nel senso che valga sempre

$$f(x) = \frac{f(x^+) + f(x^-)}{2},$$

allora la (1) diventa

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\lambda)\, e^{i\lambda x}\, d\lambda$$

cambio "nome" a variabili:

$$f(-\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(t)\, e^{-it\lambda}\, dt$$

$$\boxed{2\pi f(-\lambda)} = \int_{-\infty}^{+\infty} \hat{f}(t)\, e^{-it\lambda}\, dt \boxed{= \hat{\hat{f}}(\lambda)}$$

Quindi nelle ipotesi (2) vale la tesi: $\square$
Cioè vale il .

$\boxed{\text{Teorema}}$ $\square \Rightarrow \square$

# $\boxed{\text{Calcolo } \hat{f} \text{ (altri esempi)}}$

**ES 1** $\quad f(x) = \dfrac{1}{1+x^2}$

perché $\int_0^{+\infty} e^{-x}\,dx$ converge

ovvio

Ricordando che se $g(x) = \frac{1}{2} e^{-|x|} \in L^1(\mathbb{R}) \cap C^1_{\text{tratti}}(\mathbb{R})$

allora $\hat{g}(\lambda) = \dfrac{1}{1+\lambda^2} \in L^1(\mathbb{R})$

Perché $\int_0^{+\infty} \frac{1}{1+\lambda^2}$ converge

$$\hat{f}(\lambda) = \hat{\hat{g}}(\lambda) = \ ?$$

$$\hat{\hat{g}}(\lambda) = 2\pi\, g(-\lambda) = \cancel{2}\pi \frac{1}{\cancel{2}} e^{-|-\lambda|} = \pi e^{-|\lambda|}$$

**ES 2** $\quad \boxed{f(x) = e^{-x^2}}$

$$f'(x) = -2x e^{-x^2} = -2x f(x)$$

$$f'(x) + 2x f(x) = 0 \qquad\qquad \hat{f}(\lambda) = \mathcal{F}(f)$$

$$\widehat{f'(x)} + \widehat{2i\cdot(-ix f(x))} = \hat{0}$$

$$i\lambda \hat{f}(\lambda) + 2i\left(\hat{f}(\lambda)\right)' = 0$$

$$\begin{cases} (\hat{f})' = -\dfrac{\lambda}{2}\hat{f} \\ \hat{f}(0) = \displaystyle\int_{-\infty}^{+\infty} f(t)\,dt = \sqrt{\pi} \end{cases}$$

$$\lim_{b\to+\infty}\left(\int_{-b}^{b} e^{-x^2}\,dx\right)^2 = \lim_{b\to+\infty} \int_{-b}^{b} e^{-x^2} \cdot \int_{-b}^{b} e^{-y^2} = \lim_{b\to+\infty} \int_{[-b,b]^2} e^{-(x^2+y^2)}\,dx\,dy$$

$= \ldots$ in c. polari $\ldots = \pi$

$$\frac{(\hat{f})'}{\hat{f}} = -\frac{\lambda}{2}$$

$$\left(\ln(\hat{f})\right)' = -\frac{\lambda}{2}$$

$$\ln(\hat{f}) = -\frac{\lambda^2}{4} + c$$

$$\left.\begin{aligned} \hat{f}(\lambda) &= k e^{-\frac{\lambda^2}{4}} \\ \hat{f}(0) &= \sqrt{\pi} \end{aligned}\right\} \Rightarrow \boxed{\hat{f}(\lambda) = \sqrt{\pi}\, e^{-\frac{\lambda^2}{4}}}$$

$\boxed{a>0}$

Se era $g(x) = \boxed{e^{-a^2x^2}} = e^{-(ax)^2} = f(ax)$

$$\left(f(ax)\right)^\wedge = \frac{1}{a}\hat{f}\left(\frac{\lambda}{a}\right) = \boxed{\frac{1}{a}\sqrt{\pi}\, e^{-\frac{\lambda^2}{4a^2}}}$$

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## Convoluzione

**Def.** Date $f, g \in L^1(\mathbb{R})$ definiamo

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy$$

**OSS.** $\boxed{(f*g)^\wedge(\lambda)} = \hat{f}(\lambda) \cdot \hat{g}(\lambda)$

$$\| $$

$$\int_{-\infty}^{+\infty} (f*g)(t)\, e^{-i\lambda t}\, dt =$$

$$= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f(y)\, g(t-y)\, dy \right) e^{-i\lambda t}\, dt =$$

$$\boxed{-i\lambda(t-y) - i\lambda y}$$

$$= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f(y)\, g(t-y)\, e^{-i\lambda t}\, dy \right) dt =$$

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$$= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f(y)\, e^{-i\lambda y} \cdot g(t-y)\, e^{-i\lambda(t-y)}\, dt \right) dy$$

$(t-y)'$

$$= \int_{-\infty}^{+\infty} f(y)\, e^{-i\lambda y} \left( \int_{-\infty}^{+\infty} g(t-y)\, e^{-i\lambda(t-y)}\, dt \right) dy =$$

$$= \int_{-\infty}^{+\infty} f(y)\, e^{-i\lambda y} \left( \int_{-\infty}^{+\infty} g(u)\, e^{-i\lambda u}\, du \right) dy =$$

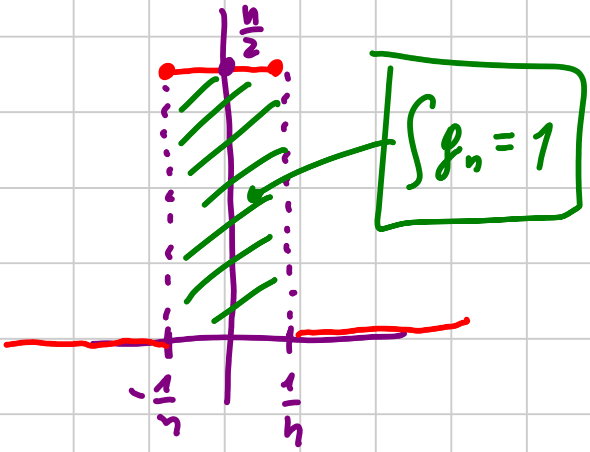
$$= \int_{-\infty}^{+\infty} g(u)\, e^{-i\lambda u}\, du \cdot \int_{-\infty}^{+\infty} f(y)\, e^{-i\lambda y}\, dy =$$

$$= \boxed{\hat{g}(\lambda) \cdot \hat{f}(\lambda)}$$

# ESEMPIO DI USO DI CONVOLUZIONE

$\boxed{\text{ES. 0}}$

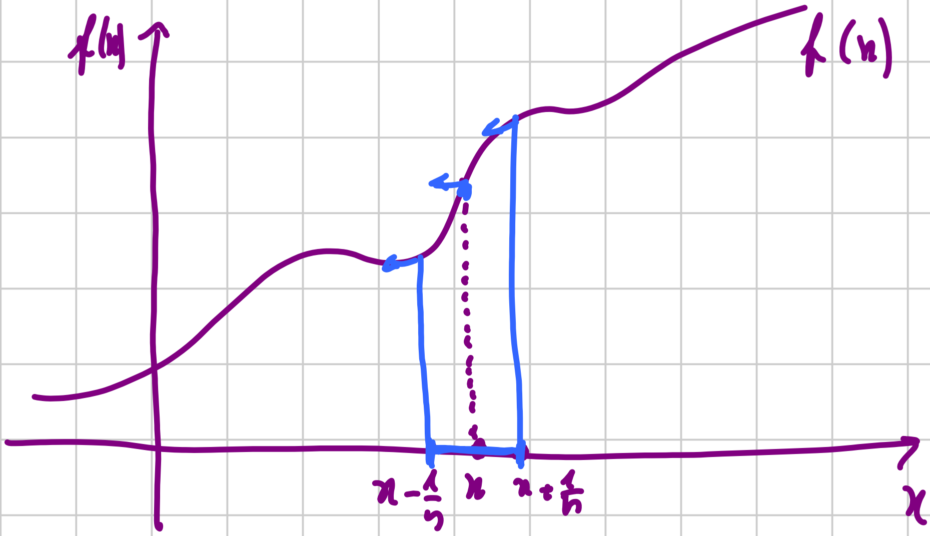
$$g_n(x) = \frac{n}{2}\chi_{[-\frac{1}{n},\frac{1}{n}]}(x)$$




$$(f*g_n)(x) = \int_{-\infty}^{+\infty} f(y)\, \overset{g_n(y-x)}{g_n(x-y)}\, dy = \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(y)\cdot\frac{n}{2}\, dy =$$

$$= \frac{1}{\frac{2}{n}}\cdot\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(y)\, dy =$$

= media integrale di $f$ in $[x-\frac{1}{n}, x+\frac{1}{n}]$




$f_n(x) = (f*g_n)(x)$

$f_n \to f$ (in qualche senso)