Metodi Matematici - Lez. 26

Titolo nota

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$\boxed{\text{Teorema 1}}$ Se $\varphi \in \mathcal{S}(\mathbb{R})$ allora $\hat{\varphi} \in \mathcal{S}(\mathbb{R})$

$\boxed{\text{DIM}}$

$$\| \lambda^m (\hat{\varphi}(\lambda))^{(k)} \|_\infty$$

$$\hat{\varphi}(\lambda) = \int_{-\infty}^{+\infty} \varphi(x) e^{-i\lambda x} \, dx$$

$$\lambda^m (\hat{\varphi}(\lambda))^{(k)} = \lambda^m \int_{-\infty}^{+\infty} (-ix)^k \varphi(x) e^{-i\lambda x} \, dx$$

$$|\lambda^m (\hat{\varphi}(\lambda))^{(k)}| = \left| \frac{1}{(-i)^m} \int_{-\infty}^{+\infty} (-ix)^k \varphi(x) \overbrace{(-i\lambda)^m e^{-i\lambda x}}^{(e^{-i\lambda x})^{(m)}} dx \right|$$

$$= \left| \frac{1}{(-i)^m} \int_{-\infty}^{+\infty} (-ix)^k \varphi(x) \left(e^{-i\lambda x}\right)^{(m)} dx \right| =$$

$$= \left| (-1)^m \frac{(-i)^k}{(-i)^m} \int_{-\infty}^{+\infty} \left(x^k \varphi(x)\right)^{(m)} e^{-i\lambda x} \, dx \right| \leq$$

$$\leq \int_{-\infty}^{+\infty} \left| \left(x^k \varphi(x)\right)^{(m)} \right| dx =$$

$$= \| (x^k \varphi(x))^{(m)} \|_{L^1} =$$

$$(f \cdot g)^{(m)} = \sum_{i=0}^{m} \binom{m}{i} f^{(i)} \cdot g^{(m-i)}$$

$$= \left\| \sum_{i=0}^{m} \binom{m}{i} (x^k)^{(i)} \cdot (\varphi(x))^{(m-i)} \right\|_{L^1} \le$$

$$\le \sum_{i=0}^{m} \binom{m}{i} \left\| (x^k)^{(i)} \cdot (\varphi(x))^{(m-i)} \right\|_{L^1} \le$$

$$\le \sum_{\boxed{\substack{i=0,\dots,k \\ j=0,\dots,m}}} C_{i,j} \left\| x^i \varphi^{(j)} \right\|_{L^1} \le$$

$$\le \underbrace{\sum_{\boxed{}} \pi C_{i,j} \left(\| x^i \varphi^{(j)} \|_{L^\infty} + \| x^{i+2} \varphi^{(j)} \|_{L^\infty} \right)}$$

$$\left| \lambda^m \left(\hat{\varphi}(\lambda)\right)^{(k)} \right| \le \dots \le \boxed{}$$

N.B. Fissati m e k la disuguaglianza vale $\forall \lambda \in \mathbb{R}$

Quindi prendendo il sup per $\lambda \in \mathbb{R}$ ottengo

$$\left\| \lambda^m \left(\hat{\varphi}(\lambda)\right)^{(k)} \right\|_{L^\infty} \le \boxed{}$$

Quindi $\hat{\varphi} \in \mathcal{S}(\mathbb{R})$

$\boxed{\text{Teo 2}}$ Se $(\varphi_n) \subset \mathcal{S}(\mathbb{R})$ e $\varphi_n \xrightarrow{\mathcal{S}(\mathbb{R})} \varphi$

allora $\hat{\varphi}_n \xrightarrow{\mathcal{S}(\mathbb{R})} \hat{\varphi}$.

$\boxed{\text{Dim}}$ Poiché $\hat{}$ è lineare basta dimostrare teorema quando $\varphi = 0$.

Procedendo come prima, per ogni $n \in \mathbb{N}$, si ha

$$\underbrace{0 \leq \left\| \lambda^m \left(\hat{\varphi}_n(\lambda)\right)^{(k)} \right\|_{L^\infty}}_{\substack{\text{per confronto} \\ \downarrow \\ 0}} \leq \underbrace{\sum_{\substack{i=0,\dots,k \\ j=0,\dots,m}} \pi C_{i,j} \left(\| x^i \varphi_n^{(j)} \|_{L^\infty} + \| x^{i+2} \varphi_n^{(j)} \|_{L^\infty} \right)}_{\substack{\text{Perché } \varphi_n \xrightarrow{\mathcal{S}(\mathbb{R})} 0 \\ \downarrow \\ 0}}$$

($C_{i,j}$ non dipendono da n)

Il fatto che per ogni fissato $m, k \in \mathbb{N}$ si abbia

$$\left\| \lambda^m \left(\hat{\varphi}_n(\lambda)\right)^{(k)} \right\|_{L^\infty} \longrightarrow 0$$

significa che $\hat{\varphi}_n \xrightarrow{\mathcal{S}(\mathbb{R})} 0$

$\boxed{\text{OSS.}}$ In $\mathcal{S}(\mathbb{R})$ vale la formula di inversione

$$\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varphi}(\lambda) e^{i\lambda x} \, d\lambda$$

$\boxed{\text{Def.}}$ Data $T \in \mathcal{S}'(\mathbb{R})$ definiamo $\hat{T}$ nel modo seguente:

$$\hat{T}(\varphi) = T(\hat{\varphi}) .$$

$\boxed{\text{OSS.}}$ La nuova def. è coerente con quella vecchia, cioè se $f \in L^1$

$$\hat{T_f} = T_{\hat{f}}$$

Infatti:

$$\hat{T_f}(\varphi) = T_f(\hat{\varphi}) = \int_{\mathbb{R}} f(x)\, \hat{\varphi}(x)\, dx = (*)$$

$$T_{\hat{f}}(\varphi) = \int_{\mathbb{R}} \hat{f}(x) \varphi(x)\, dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y) e^{-ixy} dy \right) \varphi(x)\, dx$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y) \varphi(x) e^{-ixy} dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x) e^{-ixy} dx \right) f(y)\, dy =$$

scambio per T. Fub. Tonell.

$$= \int_{\mathbb{R}} \hat{\varphi}(y)\, f(y)\, dy = (*)$$

OSS (Da fare per les. prec.)

$$\textcircled{1} = \int_{-\infty}^{+\infty} f(x)\, g'(x)\, dx = -\int_{-\infty}^{+\infty} f'(x)\, g(x)\, dx = \textcircled{2}$$

se $f \in \mathcal{S}(\mathbb{R})$ e g' e g limitate.

Perché so già che $\int$ converge

$$\textcircled{1} = \lim_{b \to +\infty} \int_{-b}^{b} f(x)\, g'(x)\, dx =$$

$$= \lim_{b\to+\infty} \left(\underbrace{\Big[f(x)\, g(x) \Big]_{-b}^{b}}_{\to 0} - \int_{-b}^{b} f'(x)\, g(x)\, dx \right) =$$

perché $f \in \mathcal{S}(\mathbb{R})$

Perché so già che $\int$ converge

$$= -\lim_{b\to+\infty} \int_{-b}^{b} f'(x)\, g(x)\, dx = -\int_{-\infty}^{+\infty} f'(x)\, g(x)\, dx$$

$$\shortparallel$$

$$\textcircled{2}$$

ES 1 $\hat{\delta}_0 = 1$

$$(\hat{\delta})(\varphi) = \delta(\hat{\varphi}) = \hat{\varphi}(0) = \int_{-\infty}^{+\infty} \varphi(x) \cdot e^{-i x 0}\, dx =$$

$$= \int_{-\infty}^{+\infty} \varphi(x)\,dx = \int_{-\infty}^{+\infty} 1 \cdot \varphi(x)\,dx = T_1(\varphi)$$

$\boxed{\text{OSS.}}$ $\hat{\hat{T}} = 2\pi\, T(-x)$ $\qquad$ $\underbrace{\hat{\hat{\varphi}} = 2\pi\, \varphi(-x)}_{\text{se } \varphi \in \mathcal{S}(\mathbb{R})}$

$$\hat{\hat{T}}(\varphi(x)) = \hat{T}(\hat{\varphi}(x)) = T(\hat{\hat{\varphi}}(x)) =$$

$$= T\big(2\pi\, \varphi(-x)\big) = 2\pi\, T(\varphi(-x)) =$$

$$= 2\pi\, \big(T(-x)\big)(\varphi(x))$$

Quindi $\hat{1} = \hat{\hat{\delta}} = 2\pi\, \delta(-x) = 2\pi\, \delta$
