Metodi Matematici - Lez. 28

Titolo nota

9 gennaio 2018 (9:30-11.15) - docente: Prof. **Emanuele Callegari** - Università di Roma ***Tor Vergata***

EQ. CALORE in $\mathbb{R}\times\mathbb{R}^+$

$$(1)\begin{cases} u_t(t,x) = u_{xx}(t,x) & (t,x)\in\mathbb{R}^+\times\mathbb{R} \\ u(0,x) = u_0(x) \leftarrow \text{suff. regolare da poter fare T.F.} \end{cases}$$

$$v(t,\lambda) = \int_{-\infty}^{+\infty} u(t,x)\, e^{-i\lambda x}\, dx = \hat{u}(t,\lambda)$$

$$v_t(t,\lambda) = \int_{-\infty}^{+\infty} u_t(t,x)\, e^{-i\lambda x}\, dx = \hat{u}_t(t,\lambda)$$

$$(1')\quad v_t(t,\lambda) = \left(u_{xx}(t,x)\right)^{\wedge}(\lambda) = (i\lambda)^2\hat{u}(t,\lambda)$$

$$= (i\lambda)^2\, v(t,\lambda)$$

$$(\star)\begin{cases} v_t(t,\lambda) = -\lambda^2 v(t,\lambda) \\ v(0,\lambda) = \hat{u}_0(\lambda) \end{cases}$$

$$\frac{v_t(t,\lambda)}{v(t,\lambda)} = -\lambda^2$$

$$\left(\ln(v(t,\lambda))\right)_t = -\lambda^2$$

$$\ln(v(t,\lambda)) = -\lambda^2 t + c$$

$$v(t,\lambda) = K e^{-\lambda^2 t}$$

$$v(0,\lambda) = \hat{u}_0(\lambda)$$

$$\hat{u}(t,\lambda) = \boxed{v(t,\lambda) = \hat{u}_0(\lambda) \cdot e^{-\lambda^2 t}}$$ sol. di $(*)$

$$\left(u(t,x)\right)^\wedge(\lambda) = \hat{u}_0(\lambda) \cdot \left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\right)^\wedge(\lambda)$$

$$\left(e^{-a x^2}\right)^\wedge(\lambda) = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{\lambda^2}{4a}} \qquad \frac{1}{4a} = t$$

$$\left(e^{-\frac{x^2}{4t}}\right)^\wedge(\lambda) = \sqrt{4t\pi} \cdot e^{-t\lambda^2}$$

$$\left(\frac{1}{\sqrt{4t\pi}} e^{-\frac{x^2}{4t}}\right)^\wedge(\lambda) = e^{-t\lambda^2}$$

$$\left(u(t,x)\right)^\wedge(\lambda) = \hat{u}_0(\lambda) \cdot \left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\right)^\wedge(\lambda)$$

$$\boxed{u(t,x)} = u_0(x) * \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} =$$

$$= \boxed{\int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}}_{G(t,y)} u_0(x-y)\, dy}$$

$\boxed{\text{OSS.}}$

$$\begin{cases} G_t(t,x) = G_{xx}(t,x) & (1) \\ G(0,x) = \delta_0 & (2) \end{cases}$$

(1)

$$G_{xx}(t,x) = \left(\frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}} \right)_{xx} =$$

$$= \frac{1}{\sqrt{4\pi t}} \cdot \left(\left(-\frac{1}{\cancel{4}_2 t} \cdot \cancel{2}x \right) \cdot e^{-\frac{x^2}{4t}} \right)_x =$$

$$= \frac{1}{\sqrt{4\pi t}} \cdot \left(-\frac{1}{2t} \cdot e^{-\frac{x^2}{4t}} + \left(-\frac{x}{2t} \right)^2 \cdot e^{-\frac{x^2}{4t}} \right) =$$

$$= \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}} \left(-\frac{1}{2t} + \frac{x^2}{4t^2} \right)$$

$$G_t(t,x) = \left(\frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}} \right)_t =$$

$$= -\frac{1}{2} \cdot \frac{1}{\sqrt{4\pi}} \cdot \frac{1}{t\sqrt{t}} \cdot e^{-\frac{x^2}{4t}} + \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}} \cdot \left(+ \frac{x^2}{4} \cdot \frac{1}{t^2} \right) =$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \left(-\frac{1}{2t} + \frac{x^2}{4t^2} \right)$$

(2) $\forall t > 0 \quad \int_{\mathbb{R}} G(t,x)\, dx = 1$

$$\int_{\mathbb{R}} G(t,x)\, dx = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}\, dx =$$

$$y = \frac{x}{\sqrt{4t}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\left(\frac{x}{\sqrt{4t}}\right)^2} \cdot \frac{1}{\sqrt{4t}}\, dx =$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2}\, dy = \frac{1}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}} = 1$$

② Per mostrare che $G(t,x) \xrightarrow{t\to 0^+} \delta_0$

devo mostrare che

$$T_{G(t,x)} \xrightarrow{t\to 0^+} \delta_0$$

cioè che $\forall \varphi \in \mathcal{D}$ si ha

$$T_{G(t,x)}(\varphi) \xrightarrow{t\to 0^+} \delta_0(\varphi) = \varphi(0)$$

$$T_{G(t,x)}(\varphi) = \int_{\mathbb{R}} G(t,x)\varphi(x)\, dx =$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \varphi(x)\, dx =$$

$$y = \frac{x}{\sqrt{4t}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\left(\frac{x}{\sqrt{4t}}\right)^2} \varphi\left(\frac{x}{\sqrt{4t}} \cdot \sqrt{4t}\right) \frac{1}{\sqrt{4t}}\, dx =$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \varphi(\sqrt{4t}\cdot y)\, dy =$$

$$\lim_{t\to 0^+} T_{G(t,x)}(\varphi) = \lim_{t\to 0^+} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \varphi(\sqrt{4t}\cdot y)\, dy =$$

Portando al lim. sotto segno integrale

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \varphi(0)\, dy =$$

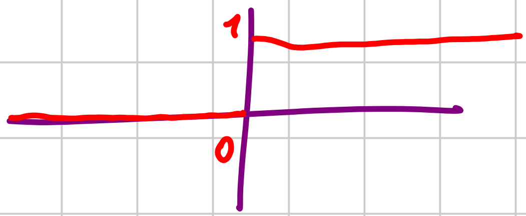
POSSO PER IL T. CONV. DOMINATA PERCHÉ $\forall t>0$

$$e^{-y^2}\varphi(\sqrt{4t}\,y) \le e^{-y^2} \cdot \|\varphi\|_{L^\infty}$$

$$= \frac{1}{\cancel{\sqrt{\pi}}}\varphi(0) \cdot \cancel{\sqrt{\pi}} = \varphi(0) .$$

$\boxed{\text{ES.1.}}$

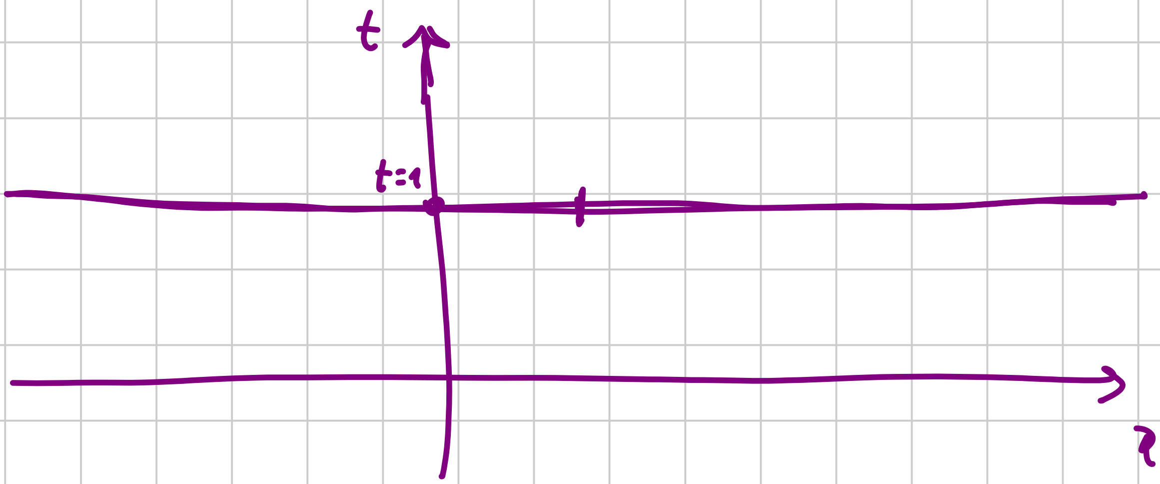
$u_0 =$



$= \chi_{[0,+\infty)}(x)$

$$u(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}} \chi_{[0,+\infty)}(\overbrace{x-y}^{>0})\, dy \qquad \boxed{x-y>0} \quad x>y \quad y<x$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}\, dy$$



$t = 1$

$\boxed{\text{ES. 2}}$ Trovare sol. $u(x)$ Fourier Trasformabile di:

$$
\begin{cases}
(1) & \displaystyle\int_{\mathbb{R}} u(y)\, u'(x-y)\, dy - \int_{\mathbb{R}} u''(y)\, u'(x-y)\, dx = -e^{-|x|}\,\mathrm{sgn}(x) \\
(2) & u(0) = -\dfrac{1}{\sqrt{2}}
\end{cases}
$$

$$\boxed{v = \hat{u}}$$

$$(1') \quad v(\lambda)\cdot\cancel{(i\lambda)}\, v(\lambda) - (i\lambda)^2 v(\lambda)\cdot\cancel{(i\lambda)}\cdot v(\lambda) = \frac{2\cancel{i\lambda}}{1+\lambda^2}$$

$$(1+\lambda^2)\,(v(\lambda))^2 = \frac{2}{1+\lambda^2}$$

$$\left(e^{-|x|}\right)^{\wedge}(\lambda) = \frac{2}{1+\lambda^2}$$

$$(v(\lambda))^2 = \left(\frac{\sqrt{2}}{1+\lambda^2}\right)^2 = \left(\frac{1}{\sqrt{2}}\cdot\frac{2}{1+\lambda^2}\right)^2$$

$$v(\lambda) = \pm\frac{1}{\sqrt{2}}\cdot\frac{2}{1+\lambda^2}$$

$$u(x) = \pm\frac{1}{\sqrt{2}}\, e^{-|x|} = -\frac{1}{\sqrt{2}}\, e^{-|x|} \qquad u(0) = -\frac{1}{\sqrt{2}}$$