Analisi Matematica 1 - Lista n. 7

Il numero e

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Calcolare i seguenti limiti:

1) $\lim\limits_{n\to+\infty}\left(1+\frac{2}{n}\right)^{n} = e^2$

2) $\lim\limits_{n\to+\infty}\left(1-\frac{3}{n}\right)^{n} = \frac{1}{e^3}$

3) $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^{n} = 1$

4) $\lim\limits_{n\to+\infty}\left(1-\frac{1}{n^2}\right)^{n} = 1$

5) $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n^2} = +\infty$

6) $\lim\limits_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n^2} = 0$

7) $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n+2}\right)^{n} = e$

8) $\lim\limits_{n\to+\infty}\left(1-\frac{1}{n+\frac{1}{n}}\right)^{n+1} = \frac{1}{e}$

9) $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^{n^4} = +\infty$

10) $\lim\limits_{n\to+\infty}\left(1-\frac{1}{n^3}\right)^{n^2} = 1$

11) $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^{2n^2+5} = e^2$

12) $\lim\limits_{n\to+\infty}\left(1+\frac{7}{n}\right)^{\frac{3n^2+5}{4n+1}} = \sqrt[4]{e^{21}}$

13) $\lim\limits_{n\to+\infty}\left(1-\frac{1}{n^2}\right)^{\frac{n^4+n+1}{n^2+7}} = \frac{1}{e}$

14) $\lim\limits_{n\to+\infty}\left(\frac{n+9}{n+5}\right)^{3n} = e^{12}$

15) $\lim\limits_{n\to+\infty}\left(\frac{n+5}{n+8}\right)^{\frac{n}{4}} = \frac{1}{\sqrt[4]{e^3}}$

16) $\lim\limits_{n\to+\infty}\left(\frac{n^2+2n+8}{n^2-3n+7}\right)^{n} = e^5$

17) $\lim\limits_{n\to+\infty}\left(\frac{n^2+n-6}{n^2+n+7}\right)^{n} = 1$

18) $\lim\limits_{n\to+\infty}\left(\frac{n^2-2n+1}{n^2-4n+7}\right)^{\frac{n^4-n+3}{2n^2+5}} = +\infty$

19) $\lim\limits_{n\to+\infty}\left(\frac{n^2-n-6}{n^2-n+2}\right)^{\frac{n^4+n+3}{n^2-1}} = \frac{1}{e^8}$

20) $\lim\limits_{n\to+\infty}\left(1+\log_n 2\right)^{\ln n} = 2$

21) $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n!}\right)^{n^n} = +\infty$

22) $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n^n}\right)^{(n+1)^n} = e^e$

23) $\lim\limits_{n\to+\infty}\frac{e^n}{\left(1+\frac{1}{n^2}\right)^{n^3}} = 1$

24) $\lim\limits_{n\to+\infty}\frac{e^{n^2}}{\left(1+\frac{1}{n}\right)^{n^3}} = +\infty$

IN PARTICOLARE ANCHE I LIMITI (23) E (24) POSSONO ESSERE CALCOLATI UTILIZZANDO SOLO METODI ELEMENTARI (CIOÈ SENZA RICORRERE ALLO SVILUPPO DELLA FUNZIONE $\ln(1+x)$, CHE NEL CORSO VERRÀ TRATTATO SOLO PIÙ AVANTI).

NELLA PAGINA SEGUENTE NE RIPORTIAMO LE SOLUZIONI (SINTETICHE!):

SOLUZIONE DI 23

$$1 = \frac{\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^n}{\left(1+\frac{1}{n^2}\right)^{n^3}} \leq \frac{e^n}{\left(1+\frac{1}{n^2}\right)^{n^3}} \leq \frac{\left(\left(1+\frac{1}{n^2}\right)^{n^2+1}\right)^n}{\left(1+\frac{1}{n^2}\right)^{n^3}} = \left(\frac{\left(1+\frac{1}{n^2}\right)^{n^2+1}}{\left(1+\frac{1}{n^2}\right)^{n^2}}\right)^n = \left(1+\frac{1}{n^2}\right)^n$$

PERCHÉ $\left(1+\frac{1}{n^2}\right)^{n^2} \leq e$
VISTO CHE $\left(1+\frac{1}{k}\right)^k \to e$ CRESCENDO

PERCHÉ $\left(1+\frac{1}{n^2}\right)^{n^2+1} \geq e$
VISTO CHE $\left(1+\frac{1}{k}\right)^{k+1} \to e$ DECRESCENDO

QUINDI:

$$\underbrace{1}_{\downarrow\, 1} \leq \frac{e^n}{\left(1+\frac{1}{n^2}\right)^{n^3}} \leq \left(1+\frac{1}{n^2}\right)^n = \underbrace{\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^{\frac{1}{n}}}_{\downarrow\, 1}$$

$$\left(1+\frac{1}{n^2}\right)^{n^2} \to e \qquad \frac{1}{n} \to 0$$

QUINDI $\frac{e^n}{\left(1+\frac{1}{n^2}\right)^{n^3}} \to 1$ PER IL TEOREMA DEL CONFRONTO.

SOLUZIONE DI 24

$$\frac{e^{n^2}}{\left(1+\frac{1}{n}\right)^{n^3}} \geq \frac{\left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{n^2}}{\left(1+\frac{1}{n}\right)^{n^3}} \geq \left(\frac{\left(1+\frac{1}{2n}\right)^2}{\left(1+\frac{1}{n}\right)}\right)^{n^3} = \left(\frac{n}{n+1}\cdot\frac{(2n+1)^2}{4n^2}\right)^{n^3} = \left(\frac{4n^2+4n+1}{4n^2+4n}\right)^{n^3} = \left(1+\frac{1}{4n^2+4n}\right)^{n^3}$$

PERCHÉ $e \geq \left(1+\frac{1}{2n}\right)^{2n}$
VISTO CHE $\left(1+\frac{1}{k}\right)^k \to e$ CRESCENDO

QUINDI:

$$\frac{e^{n^2}}{\left(1+\frac{1}{n}\right)^{n^3}} \geq \left(1+\frac{1}{4n^2+4n}\right)^{n^3} = \left(\left(1+\frac{1}{4n^2+4n}\right)^{4n^2+4n}\right)^{\frac{n^3}{4n^2+4n}} \longrightarrow +\infty$$

$$\left(1+\frac{1}{4n^2+4n}\right)^{4n^2+4n} \to e \qquad \frac{n^3}{4n^2+4n} \to +\infty$$

QUINDI ANCHE $\frac{e^{n^2}}{\left(1+\frac{1}{n}\right)^{n^3}} \to +\infty$ PER IL TEOREMA DEL CONFRONTO.