29- NOVEMBRE- 2021

EXE 09

23

ESONERO 24/11/21

Corso di Analisi Matematica 1, Laurea Triennale in Matematica

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Tutte le risposte vanno giustificate con dimostrazioni. Per confutare un enunciato basta esibire un controesempio, cioè un oggetto che soddisfi tutte le ipotesi ma non soddisfi la tesi.

Esercizio 1

Sia $A = \left\{x \in \mathbb{R} \mid x = \pi + \frac{1}{k}, k \in Z \setminus \{0\}\right\}$. Dimostrare o confutare i seguenti enunciati.

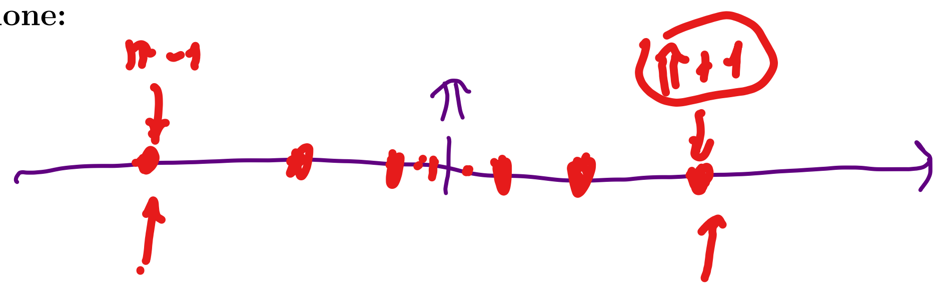
F a) $\sup A = \pi$.

F b) A non ammette massimo.

V c) $\min A = \pi - 1$.

F d) A è illimitato.

Soluzione:



Esercizio 2

Mettere in ordine di infinito crescente le seguenti successioni

$$a_n = (n^2)!, \quad b_n = n^{n(n-1)}, \quad c_n = \left(1 + \frac{1}{n!}\right)^{n^n}.$$

$\boxed{\text{SOL}}$ $\underbrace{b_n = o(a_n)}_{1}$ $\underbrace{a_n = o(c_n)}_{2}$

(1) $a_n = (n^2)! = n^2 \cdot (n^2-1) \cdot \ldots \cdot (n+1) \cdot n(n-1) \ldots 3 \cdot 2 \cdot 1 >$

$n^2 - n$

$n(n-1)$

n

n^2

$> \overbrace{n \cdot n \cdot n \ldots \cdot n}^{n(n-1)} \cdot n(n-1) \ldots 3 \cdot 2 \cdot 1 =$

$= n^{n(n-1)} \cdot n! = b_n \cdot n!$

$a_n > b_n \cdot n! \qquad \frac{a_n}{b_n} > n! \to +\infty$

(2) $c_n = \left(1 + \frac{1}{n!}\right)^{n^n} = \left(\left(1 + \frac{1}{n!}\right)^{n!}\right)^{\frac{n^n}{n!}} > \boxed{2^{n^3}}$

$a_n = (n^2)! < \boxed{(n^2)^{n^2}}$

$\frac{c_n}{a_n} > \frac{2^{n^3}}{(n^2)^{n^2}} = \left(\frac{2^n}{n^2}\right)^{n^2} \to +\infty$

$+\infty$

$\to +\infty$

$$\frac{n^n}{n!} = \underbrace{\frac{n\cdot n\cdot\ldots}{n\cdot(n-1)\ldots}}_{\geq 1}\cdot\frac{n\cdot n\cdot n\cdot n}{4\cdot 3\cdot 2\cdot 1} \geq \underbrace{\frac{n}{24}}\cdot n^3 \geq n^3$$

$n \geq 24$

Esercizio 3

Calcolare

$$\lim_{n\to+\infty}\frac{\overbrace{\ln(1+e^{n^4})}^{1}\cdot\overbrace{\ln\left(1+\frac{1}{n^2}\right)}^{2}}{\underbrace{\sqrt{n^6+n}+\sqrt{n^4+1}-n^3}_{3}} = \lim_{n\to+\infty}\frac{n^4\cdot\frac{1}{n^2}}{n^2} = 1$$

Soluzione:

$$\boxed{\frac{\ln(1+x)}{x}\to 1}$$

2 $\ln\left(1+\frac{1}{n^2}\right) \approx \frac{1}{n^2}$

1 $\ln\left(1+e^{n^4}\right) = \ln\left(e^{n^4}\cdot\left(\frac{1}{e^{n^4}}+1\right)\right) =$

$$= n^4 + \underbrace{\ln\left(1+\frac{1}{e^{n^4}}\right)}_{\to 0} \approx n^4$$

$$\sqrt{n^6+n}+\sqrt{n^4+1}-n^3 =$$

$$= \frac{\cancel{n^6}+n-\cancel{n^6}}{\underbrace{\sqrt{n^6+n}+n^3}_{\to 0}} + \underbrace{\sqrt{n^4+1}}_{\to+\infty} = \left(\underbrace{\cancel{\ }}_{\to 0}+1\right)\sqrt{n^4+1} \approx \sqrt{n^4+1} \approx n^2$$

$$\frac{\sqrt{n^4+1}}{n^2} = \sqrt{1+\frac{1}{n^4}} \to 1$$

Esercizio 4

Determinare $\alpha, \beta \in \mathbb{R}$ in modo che la funzione

$$f(x) = \begin{cases} \ln(x+\beta^2), & x > 0 \\ 1, & x = 0 \\ \frac{1-\cos(\alpha x)}{\arctan(x^2)}, & x < 0 \end{cases}$$

risulti continua nel suo dominio.

Soluzione:

$$1 = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \ln(x+\beta^2) = \ln \beta^2 \Rightarrow \beta = \pm\sqrt{e}$$

$$1 = \lim_{x \to 0^-} \frac{1-\cos(\alpha x)}{\arctan x^2} = \lim_{x \to 0^-} \frac{\frac{(\alpha x)^2}{2}}{x^2} = \boxed{\frac{\alpha^2}{2}} \Rightarrow \alpha = \pm\sqrt{2}$$

Esercizio 5

Sia $f \in \mathscr{C}(\mathbb{R}, \mathbb{R})$ una funzione tale che $f(0) = 0$. Supponiamo che esista il $\lim_{x \to 0} \frac{f(x)}{x} = \alpha \in \mathbb{R}$. Dimostrare che, dati $b > 0$ e $\beta \in \mathbb{R}$ tali che $\alpha < \beta < \boxed{\frac{f(b)}{b}}$ allora esiste $c \in]0, b[$ tale che $\frac{f(c)}{c} = \beta$.

Soluzione:

$$H(x) = \frac{f(x)}{x}$$

$$\underbrace{[0, b]}$$

$$\to H(x) = \begin{cases} \frac{f(x)}{x} & x \in (0, b] \\ \alpha & x = 0 \end{cases}$$

$$\exists c \in (0, b) \text{ t.c. } H(c) = \boxed{\beta}$$

$$\Updownarrow$$

$$\boxed{\frac{f(c)}{c}}$$

Esercizio 6

Sia $f : [0, +\infty[\to \mathbb{R}$ la funzione definita da

$$f(x) = \left(\frac{5 - \sin\left(x - \frac{1}{100}\right)}{7 + \sin x} \right)^x . \quad = 0$$

$g(x)$

Calcolarne il limite per $x \to +\infty$.

Soluzione:

$0 < A < 1$

$0 < g(x) \leq A < 1$

$$0 \leq f(x) = (g(x))^x \leq A^x \to 0$$

$$\frac{5 - \sin\left(x - \frac{1}{100}\right)}{7 + \sin x} \leq \frac{6}{6} = 1$$

$\square \leq A < 1$

$f(x_0)$

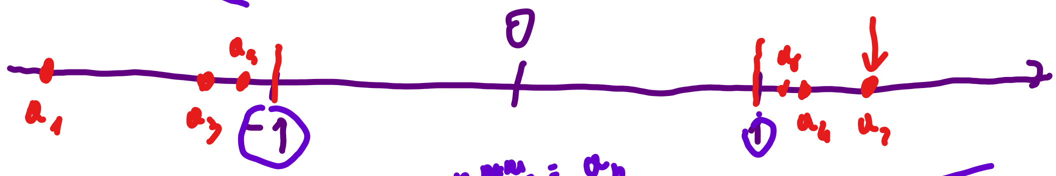
PT. DI MAX.

T. Weierstrass in $[0, 2\pi]$

CALCOLARE LIMINF E LIMSUP DI....

1) $a_n = (-1)^n\left(1+\frac{1}{n}\right)$

$$\limsup_{n\to+\infty} a_n = \lim_{n\to+\infty}\overbrace{\left(\sup_{k\geq n} a_k\right)}^{b_n}$$



$$\sup_{k\geq n} a_k = \begin{cases} n \text{ pari} = a_n \\ n \text{ dispari} = a_{n+1} \end{cases}$$

$\to 1$

2 $\quad a_n = \left(1 + \frac{(-1)^n}{n}\right)^n$ $\quad \boxed{136 \text{ LISTA } 2}$

$\boxed{n = 2k}$

$a_{2k} = \left(1 + \frac{1}{2k}\right)^{2k} \to e$ pkt. l. $\Leftarrow$ lim sup. $= e$

$\boxed{n = 2k+1}$

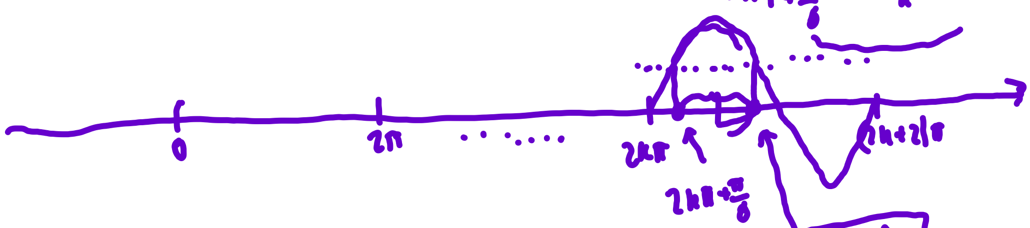
$a_{2k+1} = \left(1 - \frac{1}{2k+1}\right)^{2k+1} \to \frac{1}{e}$ pkt. l. $\Leftarrow$ lim inf $= \frac{1}{e}$

(3) $a_n = \sqrt{n}\,\overline{\sin n}$

137 LISTA 2

$\boxed{n_k}$ $\sin(n_k) > \frac{1}{2}$ $n_k = \lfloor 2k\pi + \frac{5}{6}\pi \rfloor$

$\sqrt{n_k}\sin(n_k) > \sqrt{n_k}\cdot\frac{1}{2} \to +\infty$ $2k\pi + \frac{\pi}{6} < n_k < 2k\pi + \frac{5}{6}\pi$



$a_{n_k} \to +\infty$?

$\sqrt{n_k}\cdot\sin(n_k) > \frac{1}{2}\sqrt{n_k} \to +\infty$ $\frac{2}{3}\pi > 1$
> 2

④ $a_n = \left(\sqrt{n} - \lfloor \sqrt{n} \rfloor\right)^{\sqrt{n}}$

138 LISTA 2

$n = k^2$

$a_{k^2} = 0^k = 0$

$\{x\} = x - \lfloor x \rfloor$

$\{\sqrt{1}\} = 0$

$k^2 \quad (k+1)^2$

$\{\sqrt{i}\} = 0$

$(k+1)^2 - 1 \quad \boxed{k^2 + 2k}$

$\{\sqrt{1}\} = 0$

$n = k^2 + 2k$

$a_n = \left(\sqrt{k^2+2k} - k\right)^{\sqrt{k^2+2k}} =$

$\dfrac{\sqrt{k^2+2k} - (k+1) + 1}{1} = 1 + \dfrac{-1}{\sqrt{k^2+2k} + (k+1)}$

$= \left(1 - \dfrac{1}{\sqrt{k^2+2k} + (k+1)}\right)^{\sqrt{k^2+2k}} = \left(\left(1 - \dfrac{1}{\bullet}\right)^{\bullet}\right)^{\frac{\sqrt{k^2+2k}}{\bullet}}$

$k\left(\sqrt{1+\frac{2}{k}} + 1 + \frac{1}{k}\right)$

$k\sqrt{1+\frac{2}{k}}$

$\to \frac{1}{2}$

$\to \frac{1}{2}$

$\to \frac{1}{e}$

$\frac{1}{\sqrt{e}}$

5 $a_n = \sqrt{n}\left(\sin\sqrt{n}\right)^2$

$\boxed{\text{13 9 LISTA 2}}$