Università di Roma Tor Vergata - Facoltà di ingegneria

Corso di Analisi Matematica II per ingegneria Gestionale

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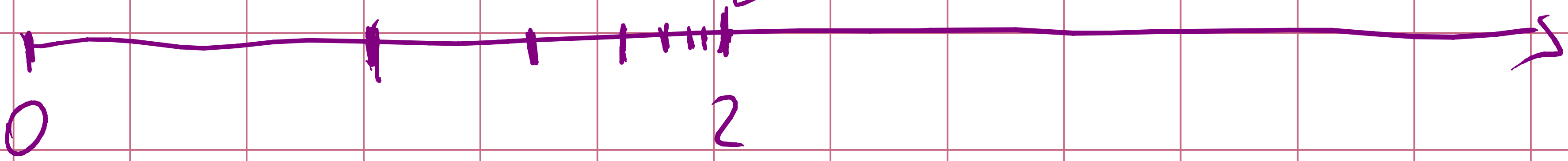
Lezione 1: SERIE NUMERICHE (I parte)

$$\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots\right) = 2 \qquad \boxed{\text{Esempio 1}}$$

(serie convergente)



$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}$$

$$\lim_{n \to +\infty} S_n = 2 \quad (?)$$

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

$$S_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^n = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \boxed{2 - \left(\frac{1}{2}\right)^n}$$

$$\lim_{n \to +\infty} 2 - \left(\frac{1}{2}\right)^n = 2$$

$\boxed{\text{Def} \mid 1}$ Data una successione $(a_n)_{n\in\mathbb{N}}$ diremo che la serie $\sum_{n=0}^{+\infty} a_n$ ad essa associata:

$$S_n \overset{\text{def}}{=} \sum_{i=0}^{n} a_i$$

→ 1) converge a $\ell \in \mathbb{R}$ se $\lim_{n\to+\infty} S_n = \ell$,

2) diverge a $+\infty$ $(-\infty)$ se $\lim_{n\to+\infty} S_n = +\infty$ $(-\infty)$.

3) indeterminata se $\lim_{n\to+\infty} S_n$ non esiste.

$\boxed{\text{Def} \mid 2}$ Diremo, invece, che $\sum_{n=0}^{+\infty} a_n$ converge assolutamente se $\sum_{n=0}^{+\infty} |a_n|$ converge

$\boxed{\text{Esempio 2}}$ $\quad \displaystyle\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ (serie di Mengoli)

$$\boxed{S_n} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots + \frac{1}{n(n+1)} =$$

$$= \frac{2-1}{1\cdot 2} + \frac{3-2}{\boxed{2\cdot 3}} + \frac{4-3}{3\cdot 4} + \frac{5-4}{4\cdot 5} + \dots + \frac{(n+1)-n}{n(n+1)} =$$

$$\frac{\cancel{3}}{2\cdot\cancel{3}} - \frac{\cancel{2}}{\cancel{2}\cdot 3}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cancel{\dots} \cancel{\dots} + \cancel{\frac{1}{n}} - \frac{1}{n+1} =$$

$$= \boxed{1 - \frac{1}{n+1}}$$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} 1 - \frac{1}{n+1} = 1$$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 \geq 1 + \frac{1}{2} + \frac{1}{2}$

$S_8 \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$\boxed{\text{Exemple 3}}$ $\quad \sum_{n=1}^{+\infty} \frac{1}{n} \quad$ $\boxed{\text{diverge à } +\infty}$

$$1 + \left(\frac{1}{2}\right) + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}_{\geq \frac{1}{2}} + \dots$$

$$\frac{1}{2^n} \quad \underbrace{\left(\frac{1}{2^n+1} + \dots + \frac{1}{2^{n+1}}\right)}_{} \geq 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}$$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 \geq 1 + \frac{1}{2} + \frac{1}{2}$

$S_8 \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$S_{16} \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$\vdots$

$S_{2^n} \geq 1 + \frac{n}{2}$

$\sum a_n$ è a termini positivi

$\downarrow$

S_n è monotona crescente

$\downarrow$

$$\lim_{n \to +\infty} \boxed{S_{2^n}} \geq \lim_{n \to +\infty} \left(1 + \frac{n}{2}\right) = +\infty \Rightarrow \lim_{n \to +\infty} S_n = +\infty$$

$\boxed{\text{Esempio 4}}$ $\sum_{n=0}^{+\infty} (-1)^n$ $\boxed{\text{indeterminata}}$

$(1-1)+(1-1)+(1-1)+(\ldots)$

$S_0 = 1$

$S_1 = 0$

$S_2 = 1$

$S_3 = 0$

$S_4 = 1$

$\left.\begin{array}{l} S_{2k+1} = 0 \\ S_{2k} = 1 \end{array}\right\} \Rightarrow S_n$ non converge

$\boxed{\text{Esempio 5}}$ $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ Convergente semplicemente ma non assolutamente

$\displaystyle\sum_{n=1}^{+\infty} a_n$ $\qquad \displaystyle\sum_{n=1}^{+\infty} |a_n| = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$.

$\boxed{\text{Osservazione}}$ Date $\sum a_n$ e $\sum \lambda a_n$ con $\lambda \in \mathbb{R} - \{0\}$

esse hanno lo stesso carattere e, nel caso convergano,

$\lambda\left(\sum a_n\right) = \sum \lambda a_n$.

$\boxed{\text{Dim}}$ Basta osservare che dette $S_n = \displaystyle\sum_{k=0}^{n} a_k$ e $\sigma_n = \displaystyle\sum_{k=0}^{n} \lambda a_n$

si la

$$\boxed{\sigma_n} = \sum_{k=0}^{n} \lambda a_k = \lambda \left(\sum_{k=0}^{n} a_k \right) = \boxed{\lambda S_n}$$

$$\sigma_n = \lambda a_0 + \lambda a_1 + \cdots + \lambda a_n = \lambda \underbrace{(a_0 + a_1 + \cdots + a_n)} = \lambda S_n$$

$$S_n \longrightarrow \ell$$

$$\sigma_n = \lambda S_n \longrightarrow \lambda \ell$$

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}$$

$$S_{2k+2} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{(2k+1)} - \frac{1}{(2k+2)}\right) =$$

$$= \frac{2-1}{1\cdot 2} + \frac{4-3}{3\cdot 4} + \frac{6-5}{5\cdot 6} + \cdots + \frac{(2k+2)-(2k+1)}{(2k+1)(2k+2)}$$

$$S_{2k+2} = \frac{1}{1\cdot 2} + 0 + \frac{1}{3\cdot 4} + 0 + \frac{1}{5\cdot 6} + 0 + \cdots + 0 + \frac{1}{(2k+1)(2k+2)}$$

$$\sigma_{2k+2} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \cdots + \frac{1}{(2k+1)(2k+2)}$$

quello che verrebbe con Mengoli

$$\boxed{S_{2k+2}} < \sigma_{2k+2}$$

$$\ell < 1$$

$$\frac{(-1)^{2k+1}}{2k+2}$$

$$S_{2k+1} = S_{2k+2} - a_{2k+2} =$$

$$\lim_{k\to+\infty} S_{2k+1} = \lim_{k\to+\infty}\left(S_{2k} + \underbrace{\frac{1}{2k+2}}_{\downarrow \atop 0}\right) = \ell + 0 = \ell$$

$\boxed{\text{Esempio 6}}$ $\displaystyle\sum_{n=0}^{+\infty} a^n$ (serie geometrica)

1) se $|a| < 1$ converge a $\frac{1}{1-a}$

2) se $a \geq 1$ diverge a $+\infty$

3) se $a \leq -1$ è indeterminata

$\boxed{\text{Caso } |a| < 1}$

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

$$S_n = \underbrace{1 + a + a^2 + \dots + a^n} = \frac{1 - a^{n+1}}{1 - a}$$

$$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a}$$

$\boxed{\text{caso } \quad a \geq 1}$

$$S_n = \begin{cases} \xrightarrow{a=1} \underbrace{1+1+1+\cdots+1}_{n+1} = n+1 \longrightarrow +\infty \\ \xrightarrow{a>1} \dfrac{1-a^{n+1}}{1-a} \longrightarrow \boxed{\dfrac{-\infty}{1-a}} = +\infty \end{cases}$$

$$\lim_{n \to +\infty} S_n = +\infty$$

$\boxed{\text{caso } \quad a \leq -1}$

$a = -1$ indeterminata

$$S_n = \frac{1-a^{n+1}}{1-a}$$ indeterminata

Esempio 7

$$\sum_{n=0}^{+\infty} \frac{1}{n!} = e = \boxed{\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n}$$

$0<\xi<x$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot x^{n+1}$$

$\boxed{0<\xi<1}$

$$\boxed{e = \underbrace{\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!}}_{S_n} + \frac{e^{\xi}}{(n+1)!}}$$

$$0 < \boxed{e - S_n} = \frac{e^{\xi}}{(n+1)!} < \frac{e}{(n+1)!}$$

$\downarrow$ $\downarrow$ $\downarrow$

0 0 0

$$\lim_{n \to +\infty} S_n = e$$

Esempio 8

$$\boxed{\sum_{n=1}^{+\infty} \frac{1}{n^2}} \quad \text{converge}$$

$$S_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

$$\sum_{n=2}^{+\infty} \frac{1}{n^2} = \boxed{\sum_{n=1}^{+\infty} \frac{1}{(n+1)^2}}$$

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$$

$$\boxed{\sigma_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}}$$

$$\boxed{S_n = \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \dots + \frac{1}{(n+1)(n+1)}}$$

$$\begin{array}{ccc} S_n & < & \sigma_n \\ \downarrow & & \downarrow \\ \ell' & & \ell \end{array}$$

Calcolare:

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n(n+1)(n+2)}$

$0,0\overline{1} \rightarrow$ 2) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{10^n}$

$0,\overline{09} \rightarrow$ 3) $\displaystyle\sum_{n=1}^{+\infty} \frac{9}{100^n}$

4) $\displaystyle\sum_{n=0}^{+\infty} \frac{n^2+2}{n!}$

5) $\displaystyle\sum_{n=3}^{+\infty} \frac{2^n+1}{n!}$

6)$^{(*)}$ $\displaystyle\sum_{n=1}^{+\infty} \frac{n}{2^n}$

Dire se converge

7) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^3}$

8) $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin \frac{n\pi}{2}}{n}$